

# Logarithmic Spiral Tilings of Triangles

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## Abstract

I describe a variety of spiral tilings of triangles in which adjacent tiles scale by a constant factor. Different ways of fitting smaller triangles to larger triangles are analyzed, and examples are given with single and multiple spiral arms. The different possibilities are explored methodically, revealing many tilings not reported previously. These tilings contain a singular point at their center where the triangles become infinitesimally small. In addition to their inherent beauty, these constructions can be used in mathematical art, sculpture, and architecture.

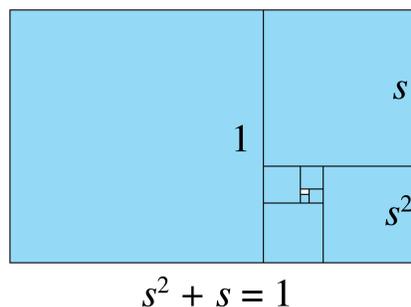
## Spiral Tilings

A spiral is a curve which emanates from a point, moving farther away as it revolves around that point. While it's difficult to precisely define what constitutes a spiral tiling, a working definition is a tiling that emanates from a finite region that the tiles move farther away from as they approximately follow a spiral curve. Spiral tilings often contain a singular point at the center. Any circular disk, however small, centered at a singular point will meet an infinite number of tiles [1].

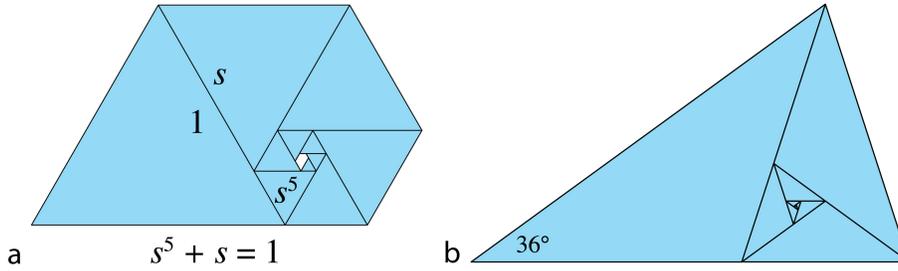
The presence of a singular point in these tilings makes them somewhat less interesting to tiling purists. On the other hands spirals have a strong esthetic resonance with people. In addition to being beautiful they're symbolic of the infinite and can create an illusion of depth. M.C. Escher incorporated spirals in a number of his prints [2].

Most spiral tilings can be described broadly as either Archimedean or logarithmic, the key difference being whether or not the tiles increase in size as distance from the center increases. The logarithmic case is more interesting and is the only one considered here. Logarithmic spiral tilings are those in which similar tiles are scaled by a constant factor. In an Archimedean spiral tiling, all the tiles are congruent. Several examples of Archimedean spiral tilings are given in Fathauer [3].

A well-known example of a logarithmic spiral tiling is the golden spiral of squares shown in Figure 1, where the scaling factor between successive squares is in the golden mean. A similar construction is possible using equilateral triangles, as shown in Figure 2a, where successive triangles scale by the plastic number, approximately 1.3247, with inverse  $\approx 0.7549$ . Also notable is the golden spiral of triangles shown in Figure 2b. In addition to the scaling factor being the golden mean, the ratio of long-to-short edges in the isosceles-triangle prototile is golden. The scaling factors for these tilings are calculated by solving simple algebraic equations obtained by labeling tile edge lengths and equating two expressions for the same distance. All of the figures in this paper depict finite portions of tilings that cover the infinite plane.



**Figure 1:** Golden spiral of squares.



**Figure 2:** a. Plastic-number spiral of equilateral triangles. b. Golden spiral of triangles.

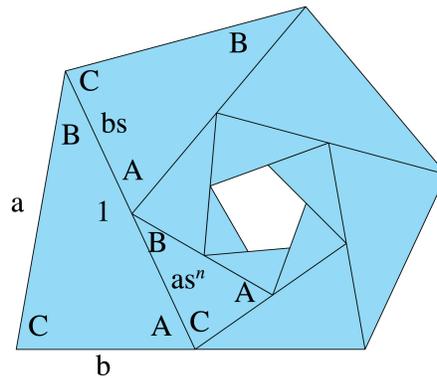
Triangles are the simplest polygons, having the least possible number of sides. While spiral tilings of quadrilaterals and other polygons are of interest, the simplest polygons are the obvious starting point for exploring spiral tilings. As shown below, the triangle case is both rich and varied.

A number of new spiral tilings were discovered in the course of this work. This was accomplished by first imagining and sketching a possible construction and then writing down and solving algebraic equations based on that sketch. Imagining the constructions requires insight and creativity, particularly for multi-armed configurations, while the algebraic equations are readily solved.

### Single-armed Spirals of Triangles

In this section, spiral tilings with a single arm will be explored. The tilings considered here have adjacent triangles sharing a vertex and mating along an edge, with successive tiles scaled by a factor  $s$ .

The general construction for a single-armed spiral tiling of scalene triangles is shown in Fig. 3 [4]. An annular patch of an infinite tiling is shown; i.e., the spiral continues *ad infinitum* both inward toward the singular point and outward to cover the entire plane. Angles and sides are labeled in the largest triangle in the patch. The length of side  $c$  (opposite angle  $C$ ) can be set to 1 without loss of generality.



**Figure 3:** a. A general one-armed spiral tiling of triangles.

Let  $n$  be the number of reduced triangles that can be mated to larger triangles until a triangle partially shares an edge with the starting triangle. In the example of Fig. 3,  $n = 5$ . We can by inspection write

$$as^n + bs = 1. \quad (1)$$

For given  $C$  and  $n$ , there are some  $A$  and  $B$  that will allow such a tiling. In order to determine  $A$ , an expression is written for the total rotation of the  $n$ th triangle to result in partially sharing an edge with the largest triangle. This triangle needs to be rotated through the angle  $A$   $n$  times to share an edge. An additional

rotation through B will bring it back to the same orientation as the largest triangle, from which we can write  $nA + B = 360^\circ$ , from which

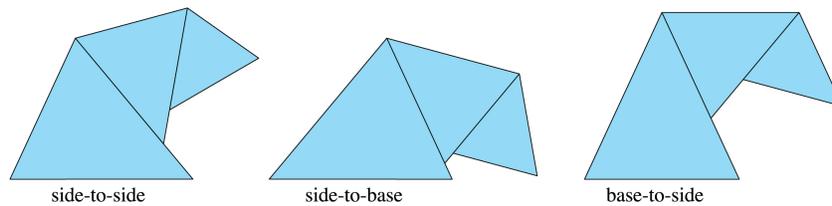
$$A = (180^\circ + C)/(n - 1), \tag{2}$$

using the fact that  $B = 180^\circ - A - C$ .

For example, in Figure 3  $n = 5$  and  $C = 80^\circ$ . From equation 2,  $A = 65^\circ$  and  $B = 35^\circ$ . The Law of Sines can then be used to calculate a and b, and equation 1 to calculate s using, e.g., an online algebraic equation solver. In the case of Figure 3,  $s \approx 0.8805$ .

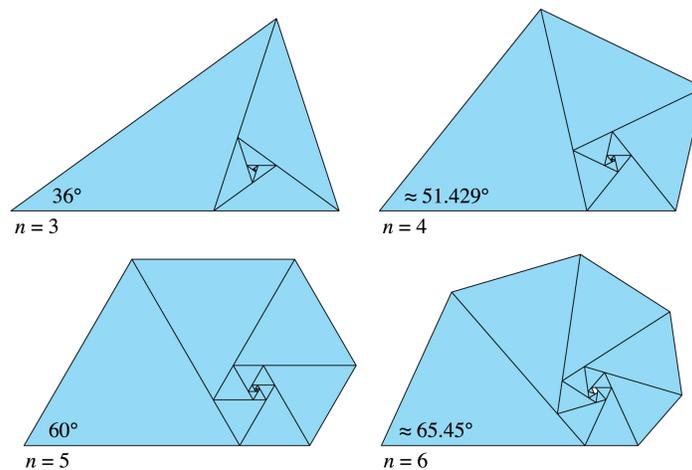
When  $A = B = C$ , equation 2 becomes  $60^\circ = 240^\circ/(n - 1)$ , indicating a spiral tiling is only admitted for  $n = 5$ . Equation 1 then gives a scaling factor satisfying  $s^5 + s = 1$ , the solution of which is the inverse of the plastic number, as seen in Fig. 2a. When  $C = 36^\circ$  and  $n = 3$ , equation 2 gives  $A = 108^\circ$ . The Law of Sines gives the golden mean for a and b, as seen in Fig. 2b.

Sticking with the idea of simple being better, equilateral, isosceles, and right triangle spiral tilings will be given special consideration. While there is a single spiral tiling of this sort using equilateral triangles, there are infinitely many spiral tilings of isosceles and right triangles. Considering the isosceles case first, there are three distinct ways of matching two isosceles triangles in such a spiral. As shown in Figure 4, these are side-to-side, side-to-base, and base-to-side.



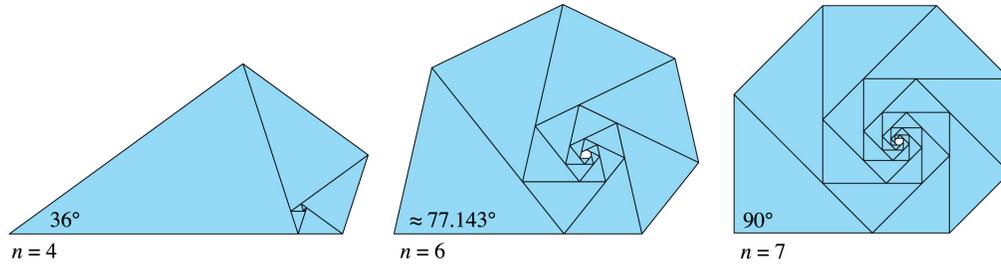
**Figure 4:** The three possibilities for mating smaller isosceles triangles to larger ones in a spiral tiling.

For side-to-side matching, a spiral tiling is allowed for any  $n > 2$ , as shown by Waldman [5]. Equation 2 simplifies to  $C = 180^\circ(n - 2)/(2n - 1)$ . The first four are shown in Figure 5, where it is seen that  $n = 3$  yields the golden triangle spiral and  $n = 5$  yields the equilateral triangle spiral.



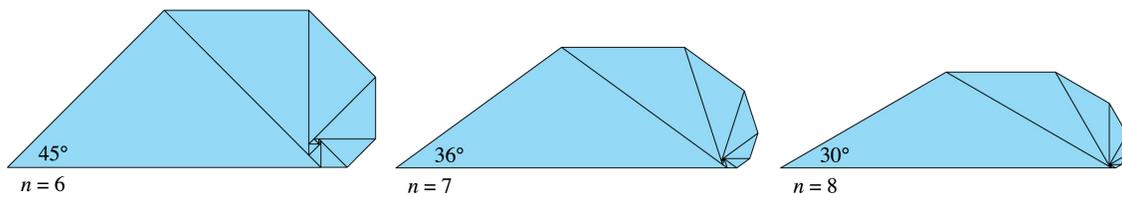
**Figure 5:** The first four spiral tilings allowed with side-to-side matching.

For side-to-base matching, a spiral tiling is allowed for any  $n > 3$ . Equation 2 simplifies to  $C = 180^\circ(n - 3)/(n + 1)$ . Three of the first four are shown in Figure 6;  $n = 5$  again yields the equilateral triangle spiral.



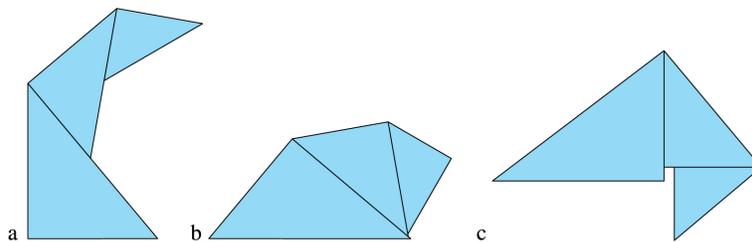
**Figure 6:** Three of the spiral tilings allowed with side-to-base matching.

For base-to-side matching, a spiral tiling is allowed for an  $n > 4$ . Equation 2 simplifies to  $C = 180^\circ/(n - 2)$ . Three of the first four are shown in Figure 7;  $n = 5$  again yields the equilateral triangle spiral.



**Figure 7:** Three of the spiral tilings allowed with base-to-side matching.

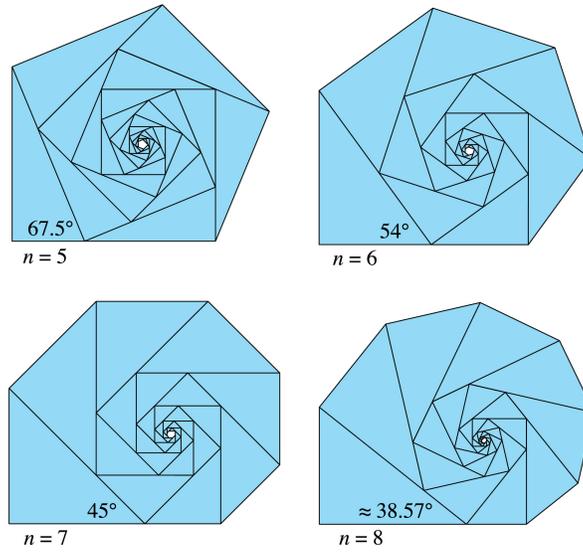
Similar to the isosceles triangle case, there are three possibilities for matching two right triangles in a tiling of this sort, as shown in Figure 8. For the first, shown in Figure 8a, equation 2 simplifies to  $A = 270^\circ/(n - 1)$ . The first four spiral tilings are shown in Figure 9.



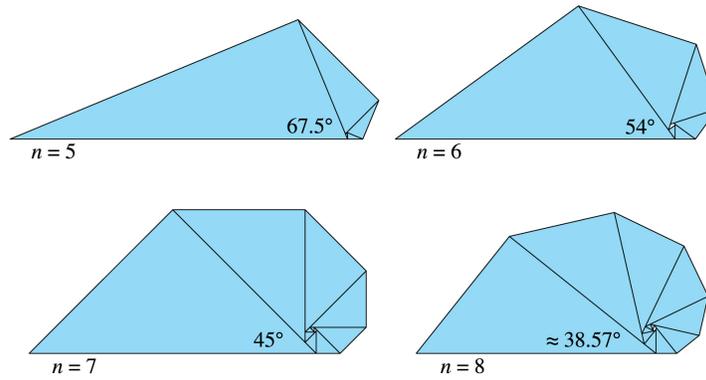
**Figure 8:** The three possibilities for mating smaller right triangles to larger ones in a spiral tiling.

For the arrangement of Figure 8b, equation 2 simplifies to  $C = 90^\circ(n - 3)/n$ . The first four spiral tilings are shown in Figure 10. Note that the same triangles admit tilings for the two right-triangle configurations. For the arrangement of Figure 8c, equation 2 simplifies to  $C = 90^\circ(n - 3)$ . There are no allowed solutions for  $C$ , which must be greater than  $0^\circ$  and less than  $90^\circ$ .

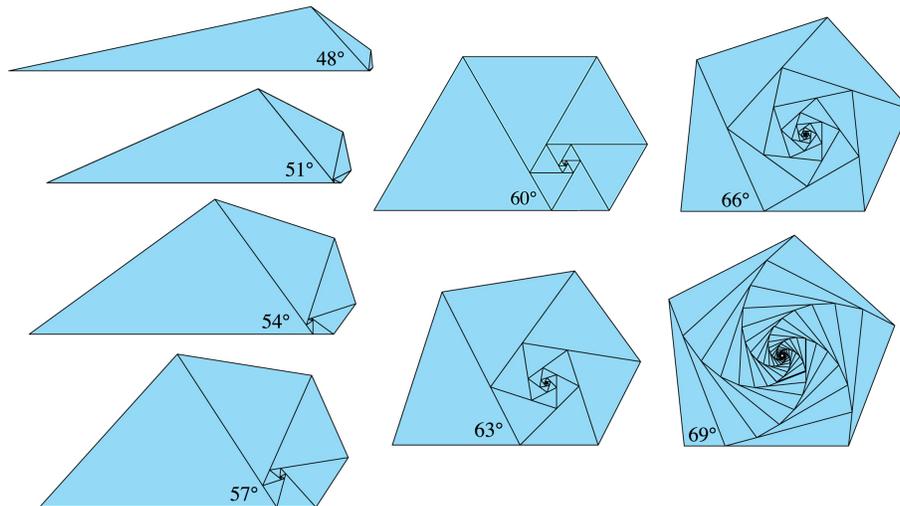
In this context there's nothing special about  $90^\circ$ . If another angle such as  $80^\circ$  were chosen instead there would still be families of allowed spiral tilings. This implies a continuous range of angles for a given  $n$ . If  $A$  is chosen as the controlling angle, then from equation 2 the angle  $A$  must be greater than  $180^\circ/(n - 1)$ , since  $C$  must be greater than  $0^\circ$ . Recall equation 2 followed from the observation that  $nA + B = 360^\circ$ . From Figure 3 it's clear that  $A$  will be at a maximum when  $B$  is at a minimum. Since  $B$  has to be greater than  $0^\circ$  the maximum value of  $A$  must be less than  $360^\circ/n$ . Therefore  $A$  must lie in the range  $180^\circ/(n - 1) < A < 360^\circ/n$ . For example, when  $n = 5$ ,  $45^\circ < A < 72^\circ$ . Figure 11 shows examples of spiral tilings with  $A$  covering most of this range.



**Figure 9:** The first four spiral tilings allowed by the matching of Fig. 8a.

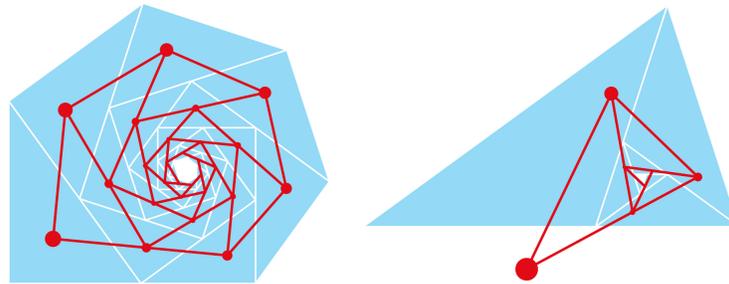


**Figure 10:** The first four spiral tilings allowed the matching of Fig. 8b.



**Figure 11:** Eight spiral tilings for  $n=5$ , with varying angles of the triangular prototile.

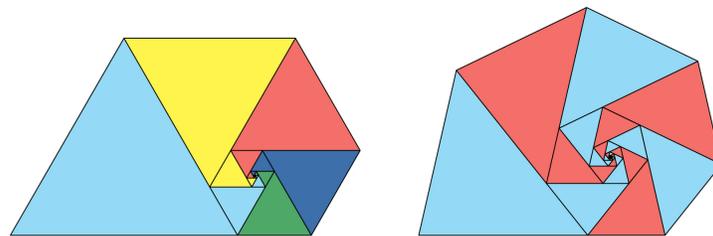
Note that the tiling formed by connecting the same point in every tile is the same as the original tiling [6]. A consequence of this is the observation that these tilings are self dual. Two examples are shown in Figure 12.



**Figure 12:** *The spiral tilings made by joining the same point in each triangle are identical to the starting tilings.*

### Multi-armed Spirals of Triangles

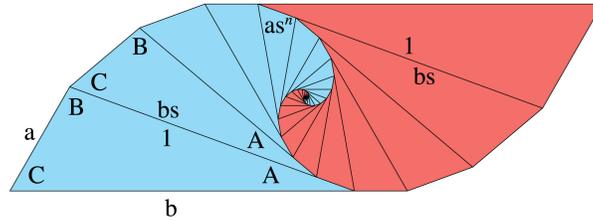
Spiral tilings can have more than one arm. In fact, all of the single-armed spirals in the preceding section are multi-armed spirals as well, with the number of arms given by  $n$ . This is obvious in the  $69^\circ$  spiral of Figure 11, where the mind strongly wants to interpret the tiling as a five-armed spiral. Note the second set of spirals in these tilings rotates opposite to the first set. All of the tilings in the preceding section have a single arm that rotates clockwise as the tiles get smaller. The multi-armed spirals defined by the same tilings rotate counter-clockwise as the tiles get smaller. Two tilings from above are shown in Figure 13 with a coloring that brings out the second set of spirals.



**Figure 13:** *Two spiral tilings from above colored to elucidate the second set of counter-rotating multi-armed spirals.*

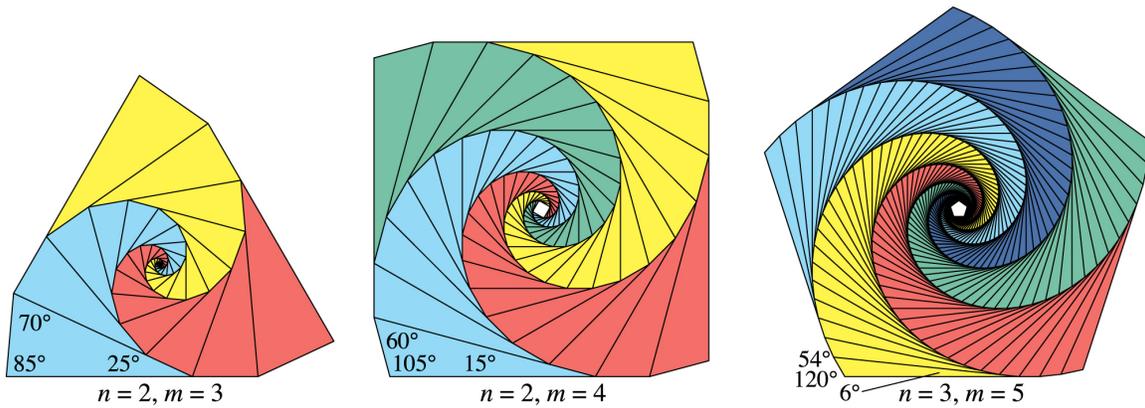
There are also spiral tilings that are multi-armed in both directions. Equation 2 above can be modified to allow multi-armed solutions, while equation 1 doesn't need modifying. Figure 14 shows a two-armed example. Recall from above that reduced tiles need to be rotated through the angle  $A$   $n$  times to share an edge with the largest triangle, with an additional rotation through  $B$  needed to bring it back to the same orientation as the adjacent large triangle, from which  $nA + B = 360^\circ$ . With multiple arms, the total angle needed to achieve the orientation of the next large triangle is  $360/m$ , where  $m$  is the number of arms, so that  $nA + B = 360^\circ/m$ . Note the relation of equation 1 still holds, with equation 2 becoming

$$A = (360/m - 180^\circ + C)/(n - 1), \quad (3)$$



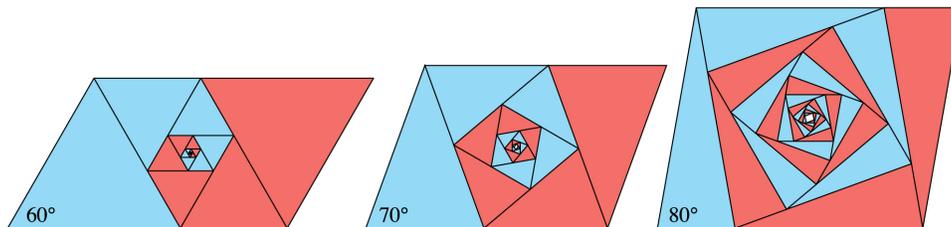
**Figure 14:** A two-armed spiral tiling used to illustrate equation 3, with 8 arms in the other direction.

Some examples are shown in Figure 15. Note the number of spirals rotating in the opposite direction is the product of  $m$  and  $n$ .



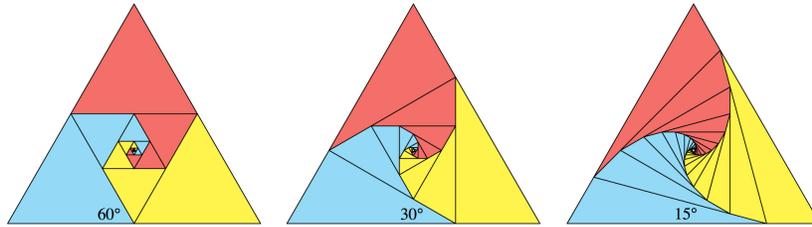
**Figure 15:** Spiral tilings with 3, 4, and 5 arms in one direction, and 6, 8, and 15 arms in the other direction.

There is a special case of equation 3 in which  $m = n = 2$ , giving  $A = C$ . This indicates an allowed spiral tiling for any isosceles triangle. A few examples are shown in Figure 16. Geometrically this works for any isosceles triangle because the envelope of a patch of next-generation-smaller tiles is a parallelogram that fits in a larger parallelogram by sharing two corners and partially sharing an edge.

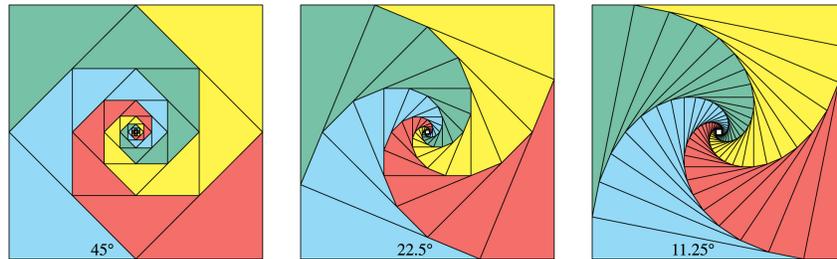


**Figure 16:** A family of two-armed tilings that can be constructed for a continuous range of side angle of the isosceles prototile.

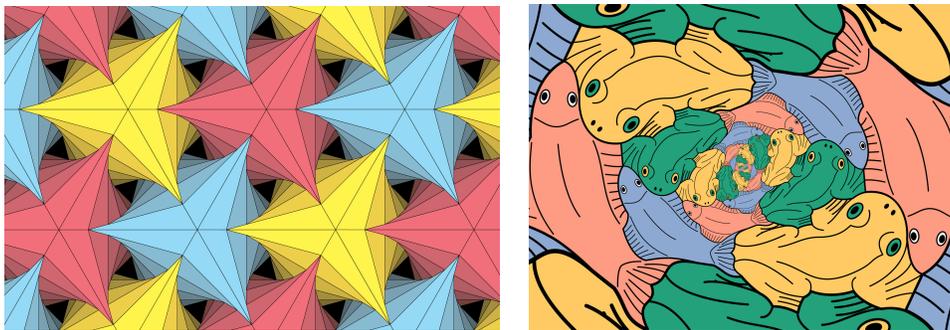
When  $n = 1$ , a regular  $m$ -gon admits  $m$ -armed spiral tilings for a continuous range of the angles  $A$  and  $B$ . The angle  $C$  fits in a corner of a regular  $m$ -gon and has the value  $180^\circ - 360^\circ/m$ . The equilateral triangle case is illustrated in Figure 17 and the square case in Figure 18. Geometrically this works because the envelope of a patch of next-generation-smaller tiles is a regular  $m$ -gon that is rotated with respect to a larger  $m$ -gon, touching at  $m$  points. A periodic tiling, based on the third tiling of Fig. 17, that is free of singular points, is shown in Fig. 19, along with an Escheresque design based on a quadruple spiral tiling of triangles.



**Figure 17:** *Three-armed spirals tilings created from regular 3-gons.*



**Figure 18:** *Four-armed spirals tilings created from regular 4-gons.*



**Figure 19:** *Periodic tessellation based on the right spiral tiling of Figure 17, and Escheresque tessellation based on a quadruple spiral of right triangles..*

## Summary and Conclusions

I have shown a wide variety of spiral tilings of triangles, many of which have not been reported previously. In addition to their mathematical interest and inherent beauty, there are numerous applications of such tilings. These include visual mathematical proofs, graphic design and mathematical art through coloring or decorating tiles (e.g., with Escheresque details or knot graphics), and sculptural and architectural forms. An area for further exploration is a systematic consideration of possibilities for spiral tilings of quadrilaterals.

## References

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