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Iterating Polyiamonds, Polyhexes, and other Polyforms to Create Fractal Reptiles

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Abstract

I extend a previously-described method for creating self-replicating (capable of being tiled by smaller copies of themselves) fractal tiles, based on iterating tessellating arrangements of polyominoes, to other polyforms. In general, while these tiles will tessellate at every stage, they will only be self replicating in the limit of an infinite number of iterations. A wide range of fractal forms result, with 2-, 3-, and 6-fold rotational symmetries, many of which have not been previously described as self-replicating tiles.

Introduction

A self-replicating tile, or "reptile", is a tile that can be tiled by smaller copies of itself, as well as tiling the plane. A simple example is an equilateral triangle, which can be tiled by four smaller triangles. There are a variety of well-known polygonal reptiles [5,6], including the triangle-based Sphinx. Denote by n the smallest number of smaller tiles a reptile can be divided into. In order to preserve area the linear scaling factor is then $\sqrt{n}$. For an equilateral triangle, $n = 4$. It can also be divided into 9 smaller triangles, 16, etc., but 4 is the smallest possible number.

In Reference 3 I described an intuitive graphical approach for generating fractal reptiles by iterative arrangements of polyominoes, shapes made up of squares connected in edge-to-edge fashion. In this paper, I apply the technique to polyiamonds (connected equilateral triangles) and polyhexes (based on regular hexagons), as well as touching on other less-common polyforms (based on other simple polygons). Several of the fractal reptiles shown here have not been described previously, to the best of my knowledge. In some cases, the fractal boundary may have been observed by other techniques such as paper folding [1], L-systems [9], or hinged Truchet tilings [10]. These constructs are related to the more general setting of substitution tilings, self-affine tiles, and fusion tilings [4, 8].

Approach

The three regular tessellations are made up of squares, equilateral triangles, and hexagons. Grids based on these provide access to a wide variety of fractal reptiles derived from polyominoes, polyiamonds, and polyhexes. These polyforms are constructed by edge-to-edge arrangement of squares, equilateral triangles, and (regular) hexagons, respectively.

The technique I employ here relies on finding linear transformations that allow grids to be mapped onto themselves. These transformations, characterized by scaling factors $\sqrt{n}$ and angles of rotation θ, can be found graphically or numerically. In Figure 1, I show grids that can be used for equilateral triangles and hexagons, with the edges of the polygons of length 1. Note that the triangle grid is unchanged by a rotation of 60° about the origin, while the hexagon grid requires a rotation of 120° to overlap itself. Reflection of grids can alternatively be used for overlaps, but rotations are easier to visualize. The values of n and θ specify the mapping of the unit vector v_1 to the vector v_2. The allowed combinations for various n can be identified using concentric circles of radius $\sqrt{n}$, as shown in Figure 1. The circles intersect the unit grid at angles that are allowed for that particular radius. Alternatively, a simple numerical approach is to look at pairs of the variables a and b shown in Figure 1. The allowed values for n through 15 are shown in Table 1 for both the equilateral triangle and hexagon cases.

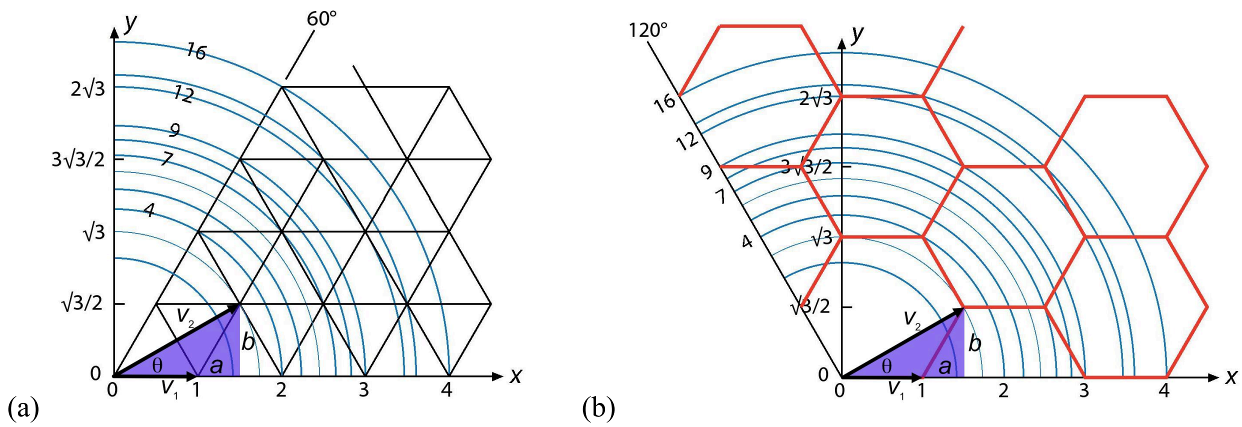


Figure 1: *Equilateral triangle (a) and hexagon (b) grids with circular arcs of various radii of magnitude √n. The arcs are labeled with their n values on the arc (a) and on the 120° line (b), and intersect the axes where x and y equal √n. The small shaded triangles illustrate the relationship between the parameters a (the magnitude of v_1), b, √n (the magnitude of v_2), and θ for transforming the vector v_1 into the vector v_2.*

Table 1: *Allowed values of n through 15 for the case of polyiamonds and polyhexes, with associated angles rounded to the nearest tenth of a degree. Points are listed for angles less than 60° for triangle-grid-only points (designated by subscript 'T') and 120° for the remaining points.*

n	3	3	4_T	4	7	7_T	9	9	12	12	13_T	13	13
$\sqrt{n}$	√3	√3	2	2	√7	√7	3	3	2√3	2√3	√13	√13	√13
a	3/2	0	2	1	5/2	2	3	3/2	3	0	7/2	5/2	1
b	√3/2	√3	0	√3	√3/2	√3	0	3√3/2	√3	2√3	√3/2	3√3/2	2√3
arctan(b/a)	30°	90°	0°	60°	19.1°	40.9°	0°	60°	30°	90°	13.9°	46.1°	73.9°

I will use positive angles to denote counterclockwise rotation from the x-axis, and negative angles to indicate clockwise rotation, with angles lying between -180° and 180°. The angles between 0° and 60° or 120° are specified in Table 1, but it should be understood that all angles satisfying the b/a ratio are included. For example, for the first column with n = 7 there are six allowed angles, with approximate values of ±19.1°, ±139.1°, and ±100.9°. Different polyforms, not all of which will result in a patch of tiles that contains neither holes nor overlaps and that additionally will tile, may result from iterating a given polyform by these different angles. See Figure 2 for the successful angles for a heptiamond case. These figures illustrate the fact that the rough overall shape of the reptile is determined by the starting polyform, while the finer structure is heavily dependent on the rotation angle.

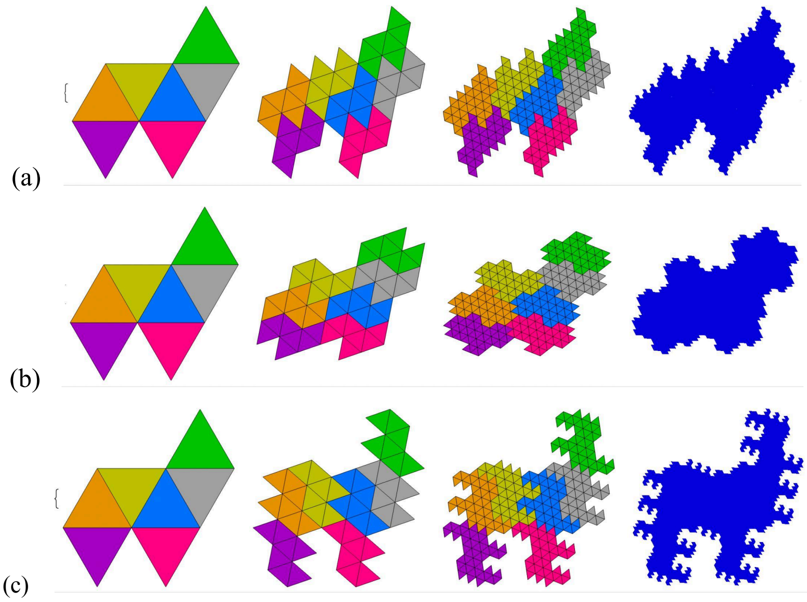


Figure 2: *The initial arrangement of seven equilateral triangles and the first three iterations in creating fractal reptiles for angles θ of approximately (a) 19.1°, (b) 139.1°, and (c) -100.9°. To elucidate the construction process in this example, each triangle is assigned a different color in the heptiamond, with the same colors used in successive iterations for sub-patches with the same position in the patch.*

The method I used to generate the reptiles is described in greater detail in Reference 3. The construction steps are as follows:

1. Select a starting polyform, consisting of n polygons, that tiles the plane. Placing a single polygon at the origin, build the polyform by a combination of translations, rotations, and/or reflections of copies of that starting polygon.
2. Scale by $1/\sqrt{n}$ about the origin and either rotate or reflect the polyform according to allowed angles from Table 1. Form the first iteration by arranging n copies of the polyform in identical fashion to that by which the polyform was built.
3. If the first iteration results in a tile without overlaps or holes that also tiles, additional iterations may be performed in the same manner. That is, repeat step 2 as many times as desired using the current iteration. In general, while the tile will tessellate at every stage, it will only be self replicating in the limit of an infinite number of iterations.

Any fractal reptiles resulting from an infinite number of such iterations can be divided into n smaller copies of itself. Each of these can in turn be divided into n smaller copies, so that an arbitrarily large patch of fractal reptiles is obtained. This is equivalent to saying the reptile can tile the infinite mathematical plane.

Polyiamond and Polyhex Examples

I present a variety of examples here, focusing on the most esthetically pleasing ones, which are often those exhibiting rotational symmetry. Examples with smaller n are also generally more interesting, as larger n means more rapid scaling between generations. Some of the examples include reflection between successive iterations.

For the purposes of this study, there are some notable differences between the three regular tessellations. While squares and equilateral triangles are themselves reptiles, hexagons are not. As a result, there are no non-fractal polyhexes that are reptiles, while polyomino and polyiamond examples do exist [6]. In addition, in order to tile the plane, equilateral triangles, unlike squares and hexagons, must appear in two different orientations. This means any iterated polyiamond will entail using two orientations of the prior-generation polyiamond. Finally, when it comes to allowed rotational symmetries, polyominoes can only possess 2- and 4-fold symmetry, while polyiamonds and polyhexes can only possess 2-, 3-, and 6-fold rotational symmetries. The respective fractal reptiles usually have the same symmetries as their generating polyforms, though there are cases such as Figure 4 in which mirror symmetry in the generating polyform does not persist in the fractal reptile. Reference 7 provides a database detailing which polyiamonds and polyhexes tile the plane and in what fashion.

Two polyiamond examples with 3-fold symmetry are shown in Figure 3. In Figure 4, two polyhex examples are shown, one with 3-fold and one with 6-fold rotational symmetry. The latter fractal is the envelope of the Gosper curve. Note in these figures how a larger value of *n* results in a fractal that is more similar to the starting polyform.

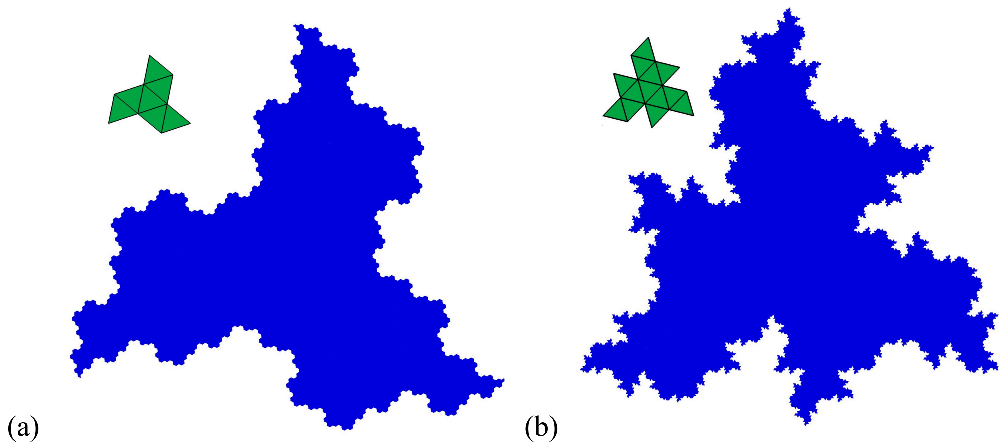


Figure 3: *Fractal reptiles formed by iterating (a) a heptiamond and (b) a 13-iamond.*

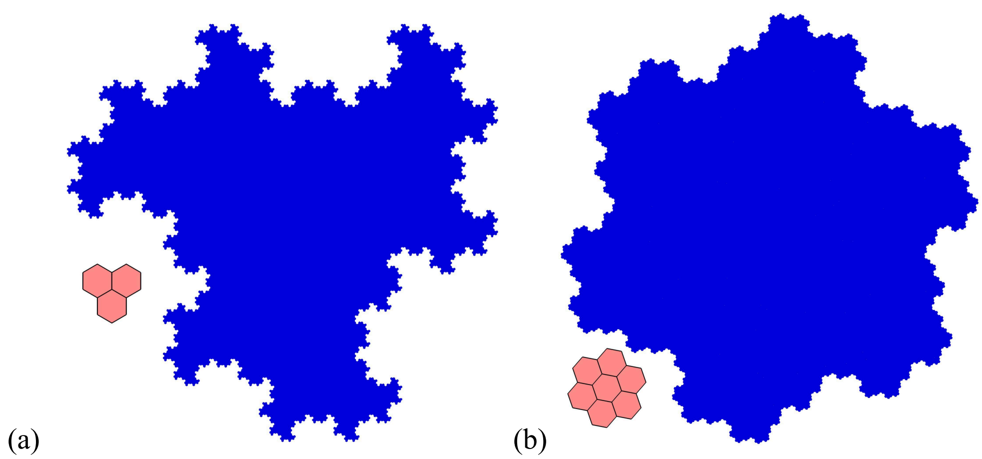
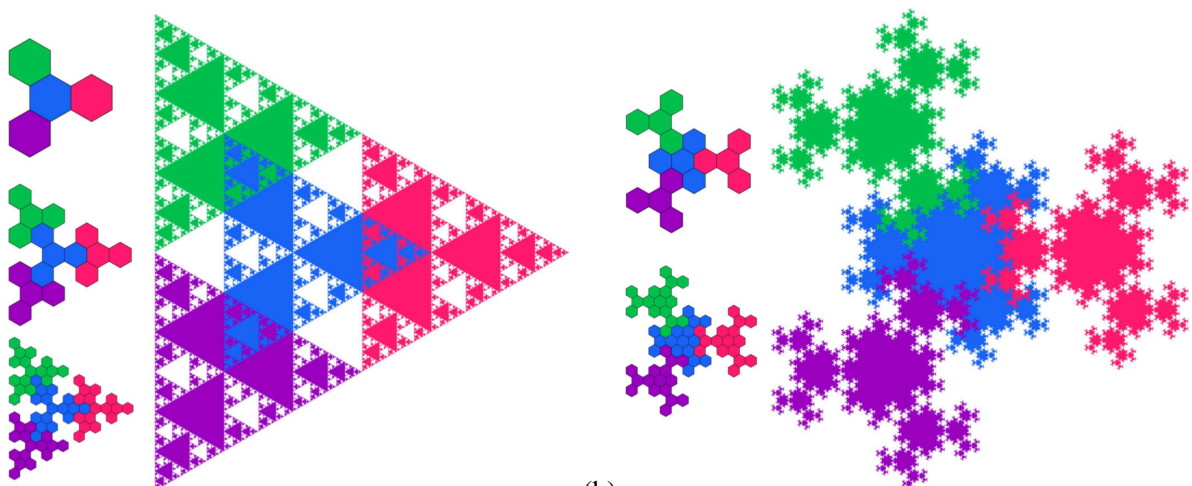


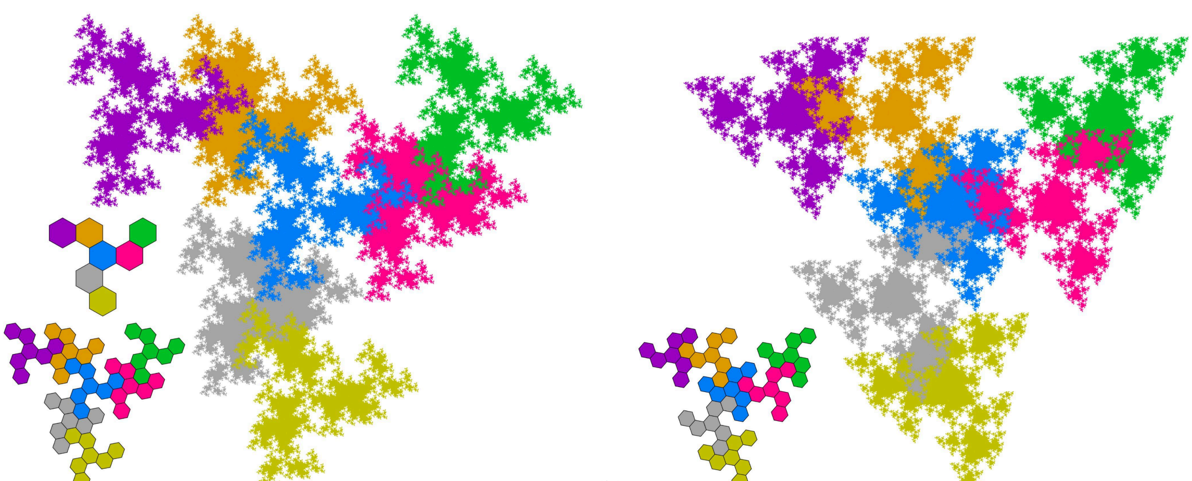
Figure 4: *(a) Reptile formed by iterating a trihex eight times. (b) Reptile formed by iterating a heptahex five times.*

In Figure 5, I show two reptiles formed by iterating a 3-fold tetrahex. In the first there is no rotation between successive iterations, while the current polyhex is rotated 60° with each iteration in Figure 5(b). Note that the starting tetrahex and resulting reptiles all possess mirror symmetry in addition to 3-fold rotational symmetry. The reptile of Figure 5(a) resembles the Sierpinski triangle, but in contrast to the Sierpinski triangle tiles the plane. In Figure 6, two reptiles formed by iterating a 3-fold heptahex are shown, with a rotation angle of approximately -19.1°. In the first there is no mirroring between successive iterations, while the current polyhex is mirrored with each iteration in Figure 6(b). On this scale, it's not clear that these complex tiles are simply connected, neither necking down to points nor having enclosed holes. On close inspection, however, one can see that there are always connections and openings of at least one constituent hexagon in size.



(a) (b)

Figure 5: *Two reptiles formed by iterating a tetrahex one, two, and seven times, with rotation angles θ of (a) 0° and (b) 60°. The starting tetrahex is inset in a.*



(a) (b)

Figure 6: *Two reptiles formed by iterating a heptahex one and five times, (a) with and (b) without mirroring between successive iterations. The starting heptahex is inset in a.*

Other Polyforms and Grids

Other grids can be used for polyform reptiles. In particular, the Laves tilings, dual to the Archimedean tilings, provide fertile ground and are the basis of several known and studied polyforms. For example, polyrhombs [2] are formed from the rhombile tessellation (with 60°-120° rhombi), and polyaboloes [2] are formed from the tetrakis square tessellation (with isosceles right triangles). Grids for these polyforms are shown in Figure 7. The rhombi appear in three different orientations, and the isosceles right triangles appear in four different orientations. Both polygons are themselves reptiles.

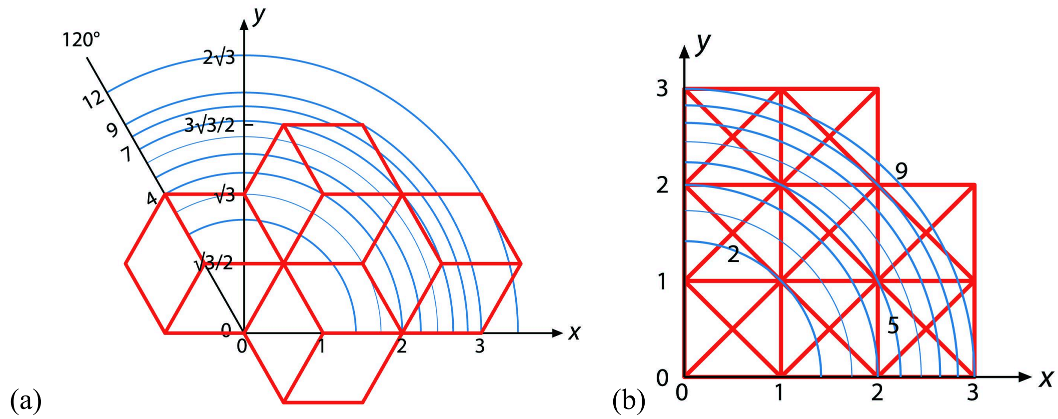


Figure 7: *Rhombus and isosceles right-triangle grids.*

In Figure 8, two reptiles formed by iterating a trirhomb are shown, using √3 scaling and an angle of 90° (see Figure 7(a)). The resulting fractal of Figure 8(a) is the envelope of the terdragon curve. In Figure 8(b) the reptile that results from mirroring between successive iterations is shown. Generally for these constructions, the choice of positioning of the origin within the grid is not unique and will affect the construction. For the trirhomb example of Figure 8 the origin was placed at the center of the middle rhombus, so that the trirhomb possesses two-fold symmetry about the origin.

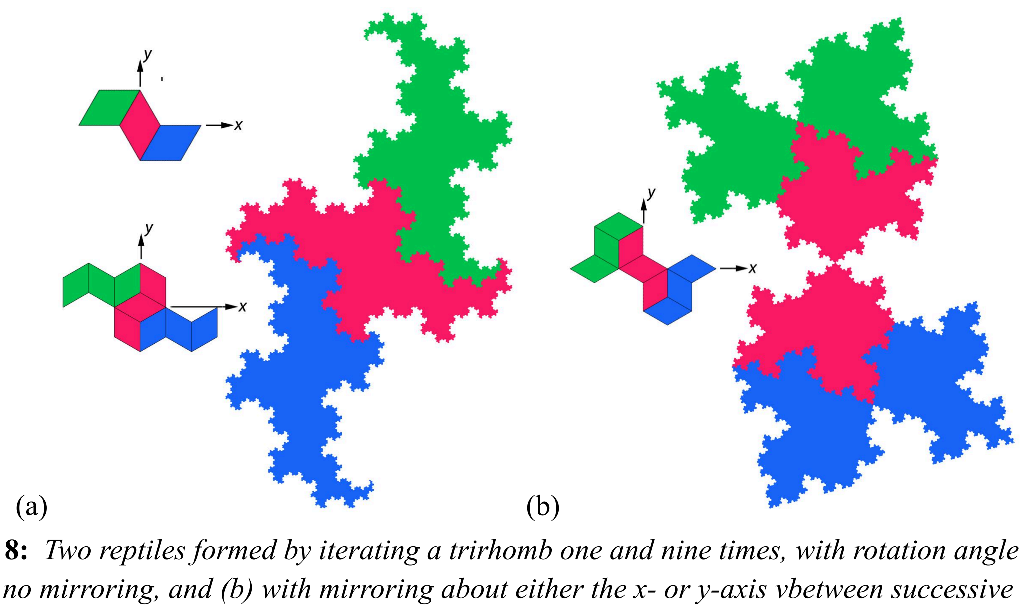


Figure 8: *Two reptiles formed by iterating a trirhomb one and nine times, with rotation angle θ of 90°; (a) with no mirroring, and (b) with mirroring about either the x- or y-axis vbetween successive iterations. The starting trirhomb is inset in a.*

In Figure 9, two reptiles formed by iterating a tetrarhomb are shown, using √4 scaling and an angle of 60° (see Figure 7(a)).

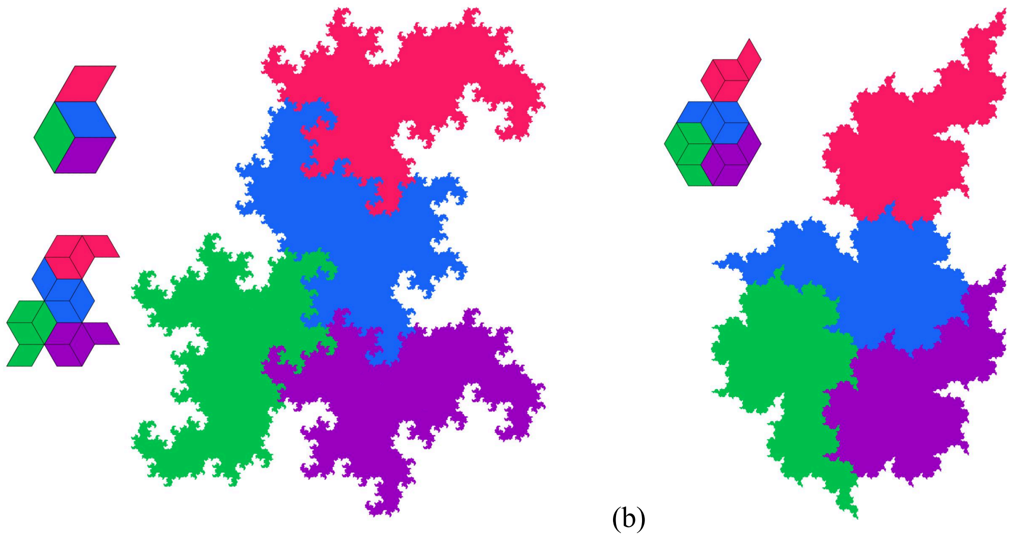


Figure 9: *Two reptiles formed by iterating a tetrarhomb one and seven times, with rotation angle θ of 60°; (a) with no mirroring, and (b) with mirroring between successive iterations. The starting tetrarhomb is inset in a.*

In addition to the positioning of the origin within the grid, the location and orientation of the starting polyform on that grid can be crucial. In Figure 10 I show two very different reptiles formed by iterating the same tetrabolo with the same rotation angle of 0°. The location and orientation of the starting tetraboloes are different, as illustrated in the figure.

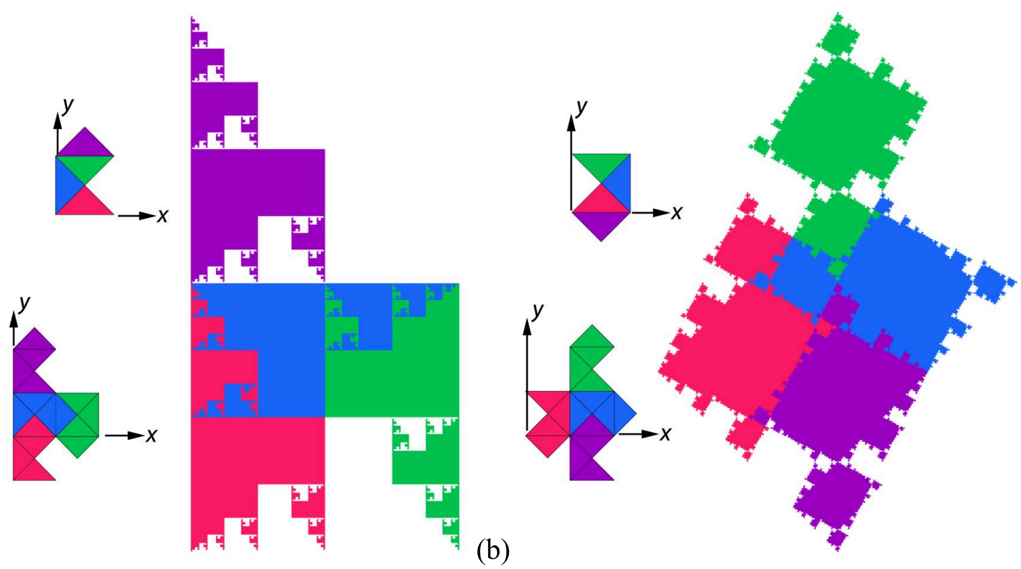


Figure 10: *Two reptiles formed by iterating the same tetrabolo with different locations and orientations of the starting tetrabolo in a and b. The first and seventh iterations are shown for each.*

As a final example, polykites [2] are created from the deltoidal hexagon tiling, as shown in Figure 11. Each kite has angles 60°-90°-120°-90°, and four are connected here to form a starting polykite.

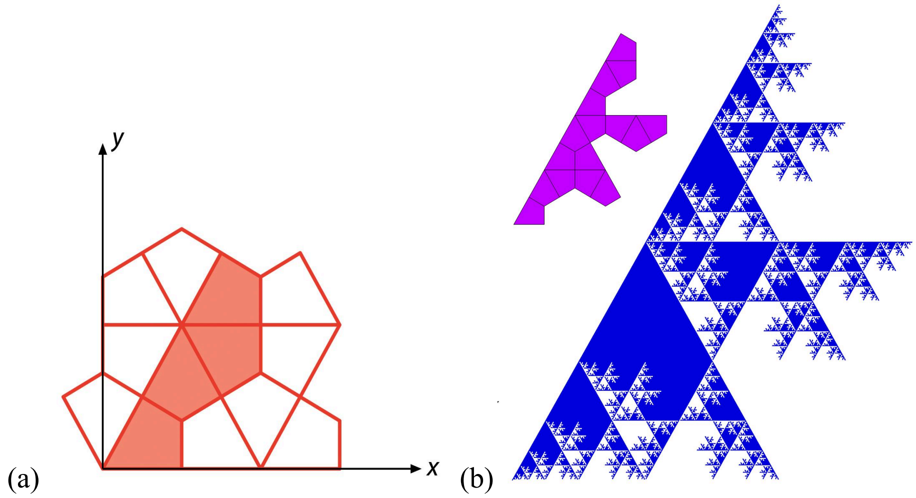


Figure 11: *(a) Polykite grid with a tetrakite. (b) A tetrakite-based reptile after one and seven iterations.*

Summary and Conclusions

I've demonstrated that iterating a variety of different polyforms can lead to fascinating and beautiful fractal self-replicating tiles. The possibilities for additional polyforms of these same and other types remain to be explored. Three-dimensional reptiles can also be constructed by this technique [3], though doing so with non-cubical building blocks would be more challenging. In the realm of mathematical art, two-dimension art could incorporate these fractals, and attractive sculptures can be formed by spatial development of fractal reptiles [3]. Escheresque tessellation art and puzzles based on these constructs could also be made.

References

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