Logarithmic Spiral Tilings of Triangles

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Abstract

I describe a variety of spiral tilings of triangles in which adjacent tiles scale by a constant factor. Different ways of mating smaller triangles to larger triangles are analyzed, and examples are given with single and multiple spiral arms. The different possibilities are explored methodically, revealing many tilings not reported previously. These tilings contain a singular point at their center where the triangles become infinitesimally small. In addition to their inherent beauty, these constructions can be used in mathematical art, sculpture, and architecture.

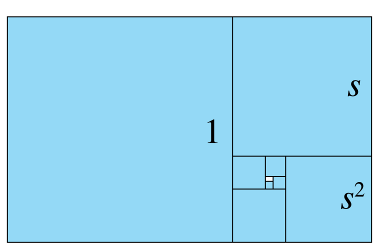
Spiral Tilings

A spiral is a curve which emanates from a point, moving farther away as it revolves around that point. While it's difficult to precisely define what constitutes a spiral tiling, a working definition is a tiling emanating along a spiral curve from a fixed point, whereby the tiles located further away from the point are larger. Spiral tilings often contain a singular point at the center. Any circular disk, however small, centered at a singular point will meet an infinite number of tiles [1].

The presence of a singular point in these tilings makes them somewhat less interesting to tiling purists. On the other hand spirals have a strong esthetic resonance with people. In addition to being beautiful, they're symbolic of the infinite and can create an illusion of depth. M.C. Escher incorporated spirals in a number of his prints [2].

Most spiral tilings can be described broadly as either Archimedean or logarithmic, the key difference being whether or not the tiles increase in size as distance from the center increases. Logarithmic spiral tilings are those in which similar tiles are scaled by a constant factor. In an Archimedean spiral tiling, all the tiles are congruent. I describe here a variety of previously unknown logarithmic spiral tilinigs of triangles. Several examples of Archimedean spiral tilings are given in References [1] and [3].

A well-known example of a logarithmic spiral tiling is the golden spiral of squares shown in Figure 1, where the scaling factor between successive squares is in the golden mean. A similar construction is possible using equilateral triangles, as shown in Figure 2(a), where successive triangles scale by the plastic number, approximately 1.3247, with inverse ≈ 0.7549 [4]. Also notable is the golden spiral of triangles shown in Figure 2(b). In addition to the scaling factor being the golden mean, the ratio of long-to-short edges in the isosceles-triangle prototile is golden. The scaling factors for these tilings are calculated by solving simple algebraic equations obtained by labeling tile edge lengths and equating two expressions for the same distance. These three spiral tilings have been known for many years, but I don't know when they were originally discovered or by whom.



$$s^2 + s = 1$$

Figure 1: *Golden spiral of squares.*

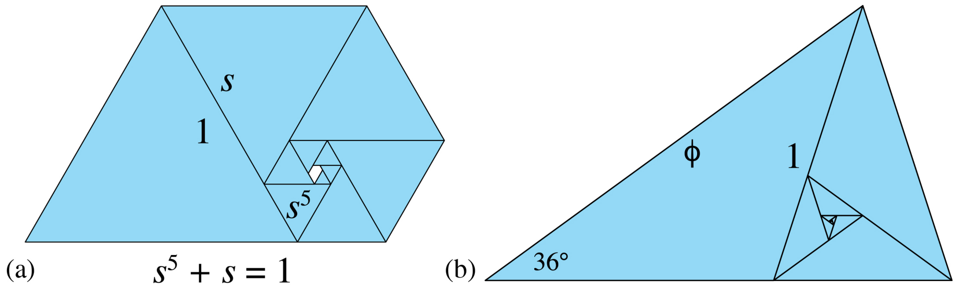


Figure 2: (*a) Plastic-number spiral of equilateral triangles. (b) Golden spiral of triangles.*

Triangles are the simplest polygons, having the least possible number of sides. While spiral tilings of quadrilaterals and other polygons are of interest, the simplest polygons are the obvious starting point for systematically exploring logarithmic spiral tilings. As shown below, the triangle case is rich and varied.

A number of new spiral tilings were discovered in the course of this work. This was accomplished by first imagining and sketching a possible construction and then writing down and solving algebraic equations based on that sketch. Imagining the constructions requires insight and creativity, particularly for multi-armed configurations, while the algebraic equations are readily solved. Previous examples of which the author is aware will be mentioned in the following as particular tilings are described. Spirals have been widely studied in general and used to model natural forms such as seashells, but such works rarely deal with tiling [5].

Single-armed Spirals of Triangles

In this section, I explore spiral tilings with a single arm. The tilings considered here have adjacent triangles sharing a vertex and mating along an edge, with successive tiles scaled by a factor s.

The general construction for a single-armed spiral tiling of scalene triangles is shown in Figure 3 [6]. An annular patch of an infinite tiling is shown; i.e., the spiral continues *ad infinitum* both inward toward the singular point and outward to cover the entire plane. Angles and sides are labeled in the largest triangle in the patch. The length of side c (opposite angle C) can be set to 1 without loss of generality.

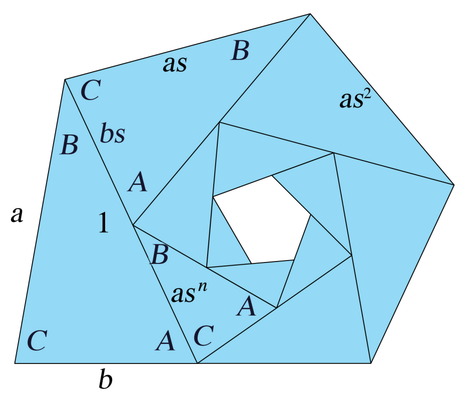


Figure 3: *General one-armed spiral tiling of triangles.*

Let n be the number of reduced triangles that can be mated to larger triangles until a triangle partially shares an edge with the starting triangle. In the example of Fig. 3, $n = 5$. We observe the relationship

$$as^n + bs = 1. \tag{1}$$

For given C and n, there are some A and B that will allow such a tiling. In order to determine A, note that the nth scaled triangle must rotate through an angle A n times in order to share an edge with the largest triangle. An additional rotation through B would bring it back to the same orientation as the largest triangle, from which we can write $nA + B = 360°$, and

$$A = (180° + C)/(n - 1), \tag{2}$$

using the fact that $B = 180° - A - C$.

For example, in Figure 3 $n = 5$ and $C = 80°$. From equation (2), $A = 65°$ and $B = 35°$. The Law of Sines can then be used to calculate a and b, and equation (1) to calculate s using, e.g., an online algebraic equation solver. In the case of Figure 3, $s \approx 0.8805$.

When $A = B = C$, equation (2) becomes $60° = 240°/(n - 1)$, indicating a spiral tiling is only admitted for $n = 5$. Equation (1) then gives a scaling factor satisfying $s^5 + s = 1$, the solution of which is the inverse of the plastic number, as seen in Fig. 2(a). When $C = 36°$ and $n = 3$, equation (2) yields $A = 108°$. The Law of Sines gives the golden mean for a, as seen in Figure 2(b).

I will give special consideration to equilateral, isosceles, and right triangle spiral tilings. While there is a single spiral tiling of this sort using equilateral triangles, there are infinitely many spiral tilings of isosceles and right triangles. Considering the isosceles case first, there are three distinct ways of mating two isosceles triangles in such a spiral. As shown in Figure 4, these are side-to-side, side-to-base, and base-to-side.

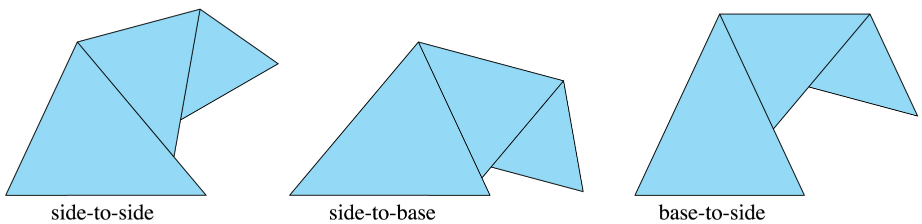


Figure 4: *The three possibilities for mating smaller isosceles triangles to larger ones in a spiral tiling.*

In the case of side-to-side matching, a spiral tiling is allowed for any $n > 2$, as shown by Waldman [7], who independently explored some of these same configurations [8]. Equation (2) simplifies to $C = 180°(n - 2)/(2n - 1)$. The first four are shown in Figure 5, where it is seen that $n = 3$ yields the golden triangle spiral and $n = 5$ yields the equilateral triangle spiral.

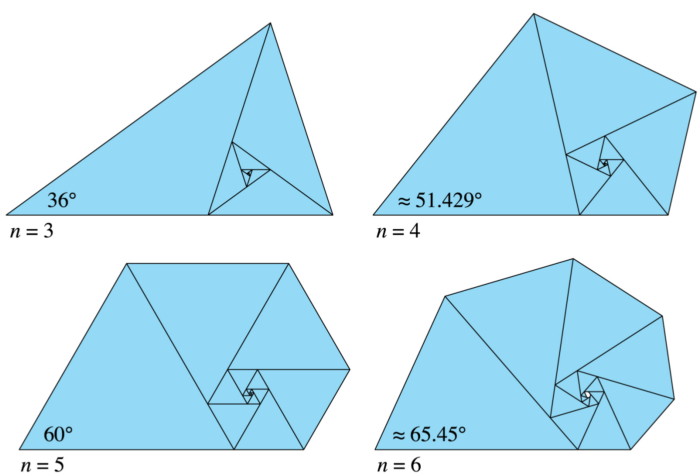


Figure 5: *The first four spiral tilings allowed with side-to-side matching, each uniquely determined by n.*

For side-to-base matching, a spiral tiling is allowed for any $n > 3$. Equation (2) simplifies to $C = 180°(n - 3)/(n + 1)$. Three of the first four are shown in Figure 6; $n = 5$ again yields the equilateral triangle spiral.

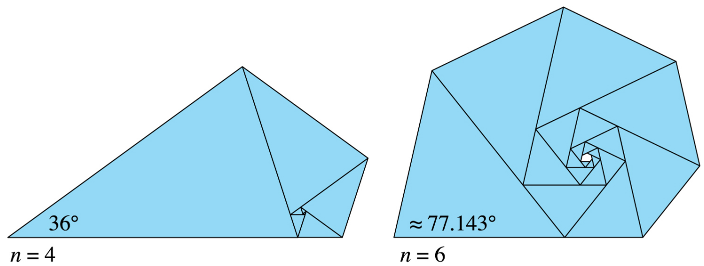


Figure 6: *Three of the spiral tilings allowed with side-to-base matching, each uniquely determined by n.*

For base-to-side matching, a spiral tiling is allowed for any $n > 4$. Equation (2) simplifies to $C = 180°/(n - 2)$. Three of the first four are shown in Figure 7; $n = 5$ again yields the equilateral triangle spiral.

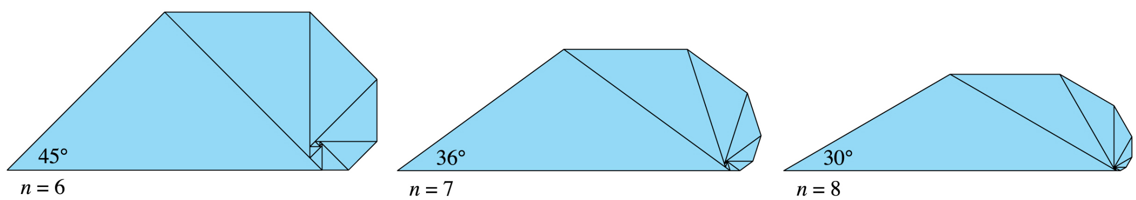
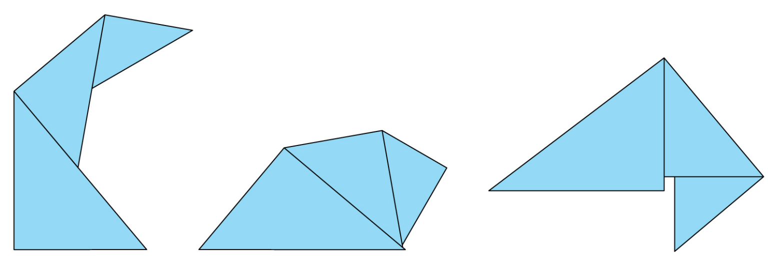


Figure 7: *Three of the spiral tilings allowed with base-to-side matching, each uniquely determined by n.*

Similar to the isosceles triangle case, there are three possibilities for mating two right triangles in a tiling of this sort, as shown in Figure 8. For the first, shown in Figure 8(a), equation (2) simplifies to $A = 270°/(n - 1)$. The first four spiral tilings are shown in Figure 9.



(a) leg to hypotenuse (b) hypotenuse to leg (c) leg to opposing leg

Figure 8: *The three possibilities for mating smaller right triangles to larger ones in a spiral tiling.*

For the arrangement of Figure 8(b), equation (2) simplifies to $C = 90°(n - 3)/n$. The first four spiral tilings are shown in Figure 10. Note that the same triangles admit tilings for the two allowed right-triangle configurations. For the arrangement of Figure 8(c), equation (2) simplifies to $C = 90°(n - 3)$, and since C must be strictly between 0° and 90°, we can't create any spirals using this arrangement.

In this context there's nothing special about 90°. If another angle such as 80° were chosen instead there would still be families of allowed spiral tilings. This implies a continuous range of angles for a given n. If A is chosen as the controlling angle, then from equation (2) the angle A must be greater than $180°/(n - 1)$, since C must be greater than 0°. Recall equation (2) followed from the observation that $nA + B = 360°$. From Figure 3 it's clear that A will be at a maximum when B is at a minimum. Since B has to be greater than 0° the maximum value of A must be less than 360°/n. Therefore A must lie in the range $180°/(n-1) < A < 360°/n$. For example, when $n = 5$, $45° < A < 72°$. Figure 11 shows examples of spiral tilings with A covering most of this range.

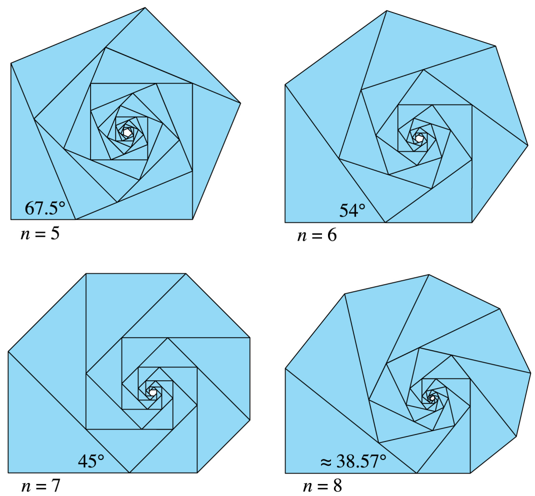


Figure 9: *The first four spiral tilings allowed by leg-to-hypotenuse matching.*

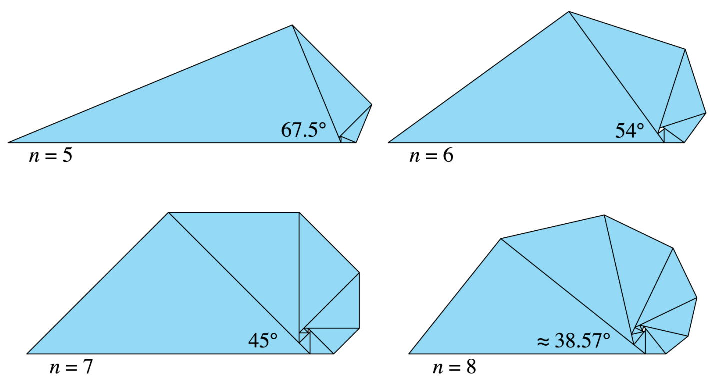


Figure 10: *The first four spiral tilings allowed by hypotenuse-to-leg matching.*

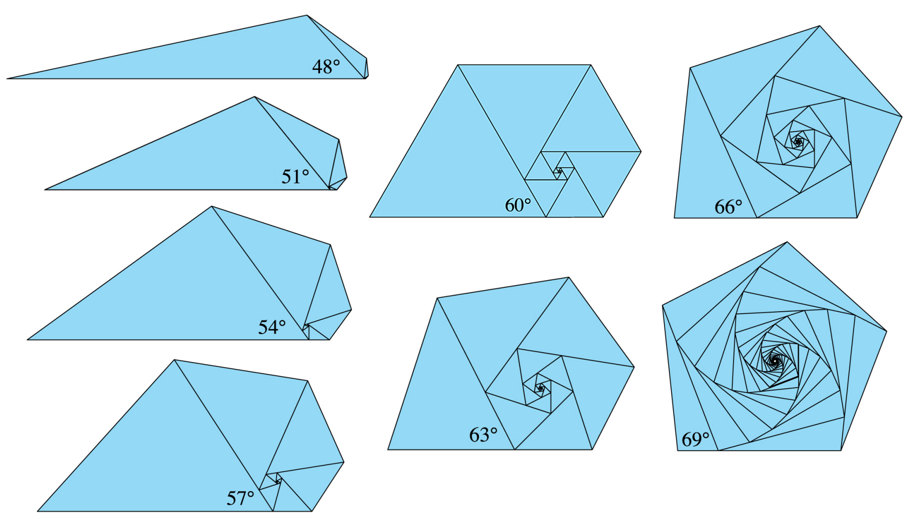


Figure 11: *Eight spiral tilings for n=5, with varying angles of the triangular prototile.*

Note that the tiling formed by connecting the same point in every tile is the same as the original tiling [8]. A consequence of this is the observation that these tilings are self dual, where the dual tiling is generated by connecting midpoints of adjacent tiles. Two examples are shown in Figure 12.

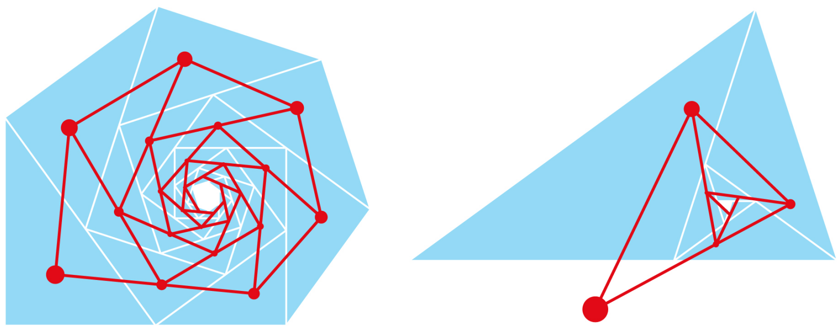


Figure 12: *The spiral tilings made by joining the same point in each triangle are identical to the starting tilings.*

Multi-armed Spirals of Triangles

Spiral tilings can have more than one arm. In fact, all of the single-armed spirals in the preceding section are multi-armed spirals as well, with the number of arms given by n. This is obvious in the 69° spiral of Figure 11, where the mind strongly wants to interpret the tiling as a five-armed spiral. All of the tilings in the preceding section have a single arm that rotates clockwise as the tiles get smaller. The multi-armed spirals defined by the same tilings rotate counter-clockwise as the tiles get smaller.

There are also spiral tilings that are multi-armed in both directions. Equation (2) above must be modified to describe these tilings, while equation (1) doesn't need modifying. Figure 13 shows a two-armed example. Recall from above that reduced tiles need to be rotated through the angle A n times to share an edge with the largest triangle, with an additional rotation through B needed to bring it back to the same orientation as the adjacent large triangle, from which $nA + B = 360°$. With multiple arms, the total angle needed to achieve the orientation of the next large triangle is $360°/m$, where m is the number of arms, so that $nA + B = 360°/m$. Note the relation of equation (1) still holds, with equation (2) becoming

$$A = (360/m - 180° + C)/(n - 1). \quad (3)$$

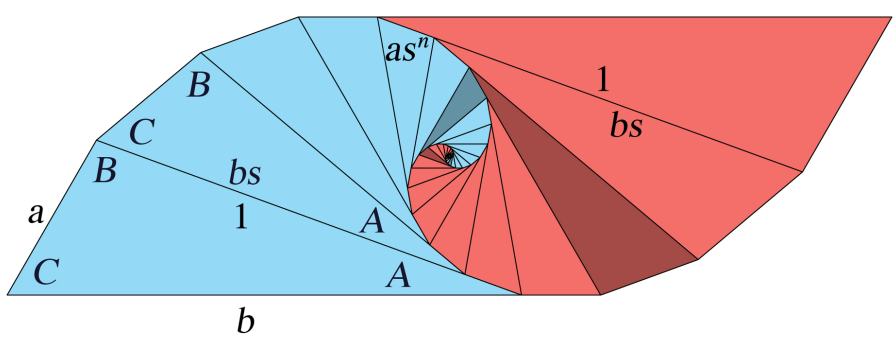


Figure 13: *A two-armed spiral tiling used to illustrate equation (3), with 8 arms in the other direction, one of which is darkened for clarity.*

Some examples are shown in Figure 14. Note the number of spirals rotating in the opposite direction is the product of m and n.

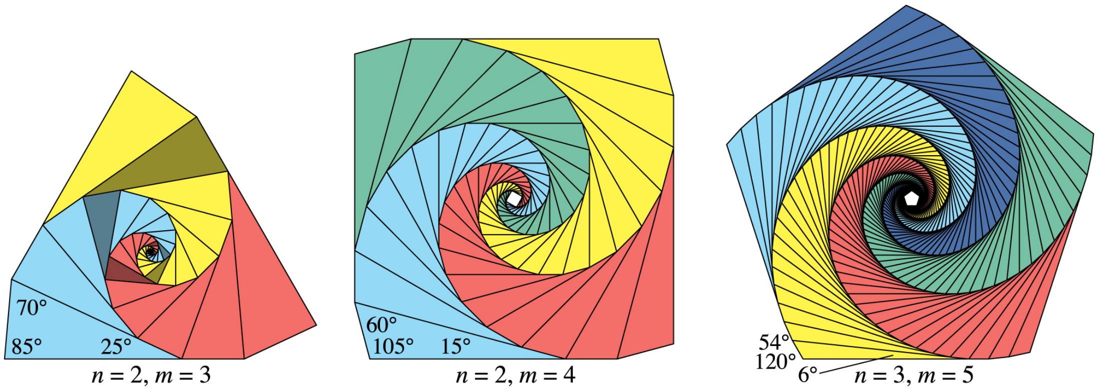


Figure 14: *Spiral tilings with 3, 4, and 5 arms in one direction, and 6 (one darkened to emphasize it), 8, and 15 arms in the other direction.*

There is a special case of equation (3) in which $m = n = 2$, giving A = C. This indicates an allowed spiral tiling for any isosceles triangle. A few examples are shown in Figure 15. Geometrically this works for any isosceles triangle because the envelope of a patch of next-generation-smaller tiles is a parallelogram that fits in a larger parallelogram by sharing two corners and partially sharing an edge.

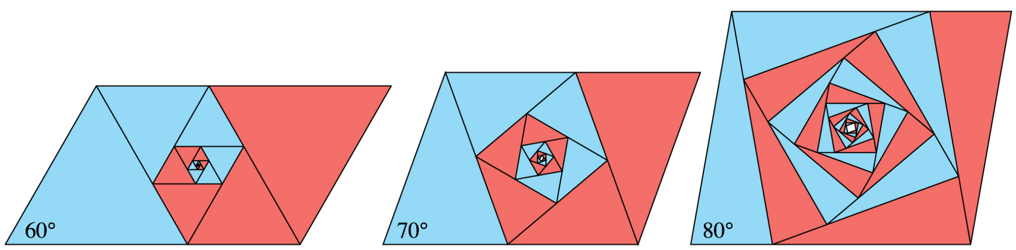


Figure 15: *A family of two-armed tilings that can be constructed for a continuous range of side angle of the isosceles prototile.*

When $n = 1$, a regular m-gon admits m-armed spiral tilings for a continuous range of the angles A and B. The angle C fits in a corner of a regular m-gon and has the value 180° – 360°/m. The equilateral triangle case is illustrated in Figure 16 and the square case in Figure 17. Geometrically this works because the envelope of a patch of next-generation-smaller tiles is a regular m-gon that is rotated with respect to a larger m-gon, touching at m points. A periodic tiling, based on the third tiling of Figure 16, that is free of singular points, is shown in Figure 18, along with an Escheresque design based on a quadruple spiral tiling of triangles. Spirals based on rotated regular polygons have been known for some time [9].

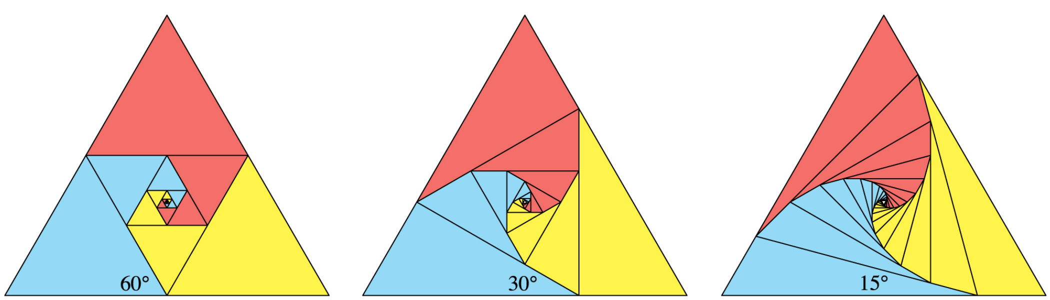


Figure 16: *Three-armed spirals tilings created from regular 3-gons.*

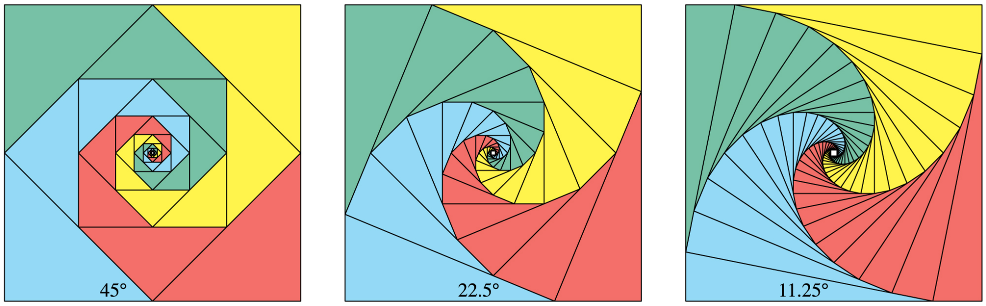


Figure 17: *Four-armed spirals tilings created from regular 4-gons.*

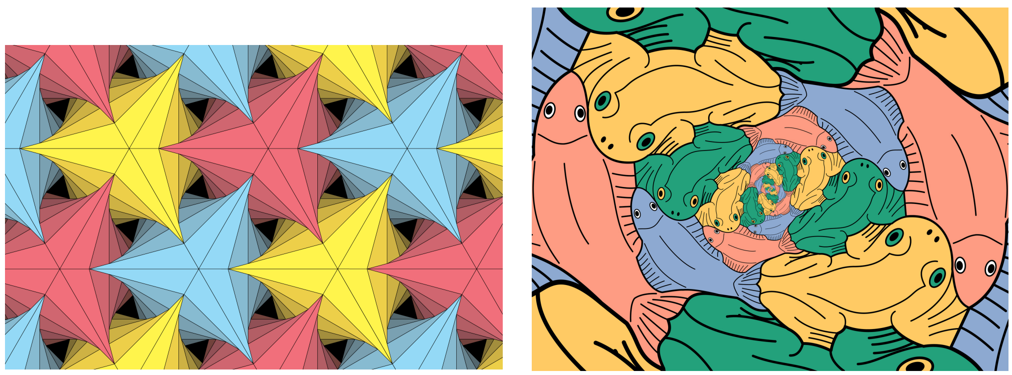


Figure 18: *Periodic tessellation based on the right spiral tiling of Figure 16, and Escheresque tessellation based on a quadruple spiral of right triangles (geometric template not included here).*

Summary and Conclusions

I have presented a wide variety of spiral tilings of triangles, many of which have not been reported previously. In addition to their mathematical interest and inherent beauty, there are numerous applications of such tilings. These include visual mathematical proofs, graphic design and mathematical art through coloring or decorating tiles (e.g., with Escheresque details or knot graphics), and sculptural and architectural forms. An area for further exploration is a systematic consideration of possibilities for spiral tilings of quadrilaterals, pentagons, etc.

References

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