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Fractal Tilings Based on Dissections of Polyominoes, Polyhexes, and Polyiamonds

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Abstract

Fractal tilings ("*f*-tilings") are described based on single prototiles derived from dissections of polyominoes, polyhexes, and polyiamonds. These prototiles have one or two long edges and two or more short edges, and the angles between long and short edges are in most cases irrational. The *f*-tilings are constructed by iterative arrangement, according to a simple matching rule, of successively smaller generations of tiles about a central group of largest-generation tiles that form the generating polyomino, polyhex, or polyiamond. For the most part, these *f*-tilings do not cover the infinite plane, but rather are bounded and contain singular points. Within their boundaries, which are in most cases fractal curves, they contain neither gaps nor overlaps, and the *f*-tilings presented here are all edge-to-edge or pseudo-edge-to-edge.

Fractals and tilings can be combined to form a variety of esthetically appealing constructs that possess fractal character and at the same time obey many of the properties of tilings. Previously, we described families of fractal tilings based on kite- and dart-shaped quadrilateral prototiles [1], v-shaped prototiles [2], and prototiles constructed from segments of regular polygons [3]. Many of these constructs may be viewed online [4]. These papers appear to be the first attempts at a systematic treatment of this topic, though isolated examples were earlier demonstrated by M.C. Escher [5] and Peter Raedschelders [6].

In Grünbaum and Shephard's book *Tilings and Patterns* [7], a tiling is defined as a countable family of closed sets (tiles) that cover the plane without gaps or overlaps. The constructs described in this paper do not for the most part cover the entire Euclidean plane; however, they do obey the restrictions on gaps and overlaps. To avoid confusion with the standard definition of a tiling, these constructs will be referred to as "*f*-tilings", for fractal tilings.

The tiles used here are "well behaved" by the criterion of Grünbaum and Sheppard; namely, each tile is a (closed) topological disk. Most of the *f*-tilings explored in References 1-4 are edge-to-edge; i.e., the corners and edges of the tiles coincide with the vertices and edges of the tilings. However, they are not "well behaved" by the criteria of normal tilings; namely, they contain singular points, defined as follows. Every circular disk, however small, centered at a singular point meets an infinite number of tiles. Since any *f*-tiling of the general sort described here will contain singular points, we will not consider singular points as a property that prevents an *f*-tiling from being described as "well behaved". These *f*-tilings provide a rich source of unique fractal images and also possess considerable recreational mathematics content.

The prototiles considered in this paper are derived by dissecting polyominoes, polyhexes, and polyiamonds. These are shapes made by connecting squares, regular hexagons, and equilateral triangles, respectively, in edge-to-edge fashion. Polyominoes made from *n* squares will be called *n*-ominoes, and similarly for polyhexes and polyiamonds, except for polyominoes made up of a small number of squares, which will be referred to using standard prefixes such as "hex" for 6.

For a discussion of different types of polyominoes, polyhexes, and polyiamonds, and conventional tilings using them, see Reference 7.

In this paper, we construct *f*-tilings from prototiles created by dissecting these shapes, as illustrated in Figure 1. A 24-iamond is shown at upper left in this figure, dissected into six congruent tiles. Note that each tile has two long edges and four short edges, where the short edges are defined by the edges of the constituent triangles in the polyiamond. (A distinction between true edges and pseudo-edges is made below.) An *f*-tiling is created by fitting smaller tiles around larger tiles, where a long edge of the smaller tile has the same length as a short edge of the larger tile. At upper right in Figure 1, the scaling factor s and angle of rotation α between successive generations of tiles are indicated. Straightforward application of trigonometry reveals that α and s are both irrational, with approximate values of 19.11° and 0.3780. At lower left, two possible matching rules for arranging the smaller tiles are marked with black dots. Matching Rule A does not allow tiling without overlaps, while Matching Rule B does. The first two iterations of the construction of the *f*-tiling using Matching Rule B are shown at lower right.

In order to generate the full *f*-tiling, the same matching rule is applied repeatedly with successively smaller generations of tiles. In theory, an infinite number of such iterations should be performed. In practice, the overall structure of the *f*-tiling changes little after roughly a half dozen iterations due to the fact that the tiles shrink rapidly. This simple process generates *f*-tilings with fractal boundaries that can be quite complex. Figure 2 illustrates this phenomenon by continuing the construction begun in Figure 1. A portion of the *f*-tiling is shown after each of the first six iterations. In this particular *f*-tiling, an intricate channel forms. It is clear from the similarity of the channel from one iteration to the next that this channel will become infinitesimally narrow with repeated iteration, but never fully close off.

In general, the boundaries of *f*-tilings are fractal curves, though there are cases in which the boundaries are non-fractal polygons. The boundaries are similar to Koch islands and related constructs, in which a line segment is distorted into multiple smaller line segments, which are in turn distorted according to the same rule, etc.

Now that the overall process is a little clearer, we can examine possible prototiles in more depth. Due to space limitations, only polyomino prototiles will be described in detail. The major features of polyhex and polyiamond prototiles will be summarized after the polyomino discussion.

Most polyominoes have adjacent straight-line segments longer than the edges of the constituent squares, as shown in Figure 3. For this reason, there are relatively few true edge-to-edge *f*-tilings using prototiles created by dissecting polyominoes. In order to provide a richer variety of examples, pseudo-edge-to-edge *f*-tilings will be considered as well. In these cases, the edges of the constituent squares are considered to define pseudo-edges of the polyominoes, as shown in Figure 3.

Each prototile has one or two long edges and two or more short edges. The angles between the long edges and adjacent short edges are for the most part irrational, but sum to a multiple of $\pi/2$, as shown in Figure 3. The *f*-tilings are constructed by first matching the long edges of identical prototiles, such that the group of first-generation tiles forms the dissected polyomino. Other more complex arrangements of first generation tiles are possible. In addition, *f*-tilings may be constructed that possess more than one prototile, but these will not be considered here.

Three requirements simplify the search for polyomino-based prototiles that allow *f*–tilings.

1. The generating polyomino must have 2-fold or 4-fold rotational symmetry. While polyominoes without rotational symmetry may be used, they generate no new prototiles, and therefore they do not need to be considered. While not proven here, this is readily apparent by examining a table showing all polyominoes possible for a given number of squares. In addition, mirrored variants of polyominoes are not considered to generate distinct prototiles, as they would result in *f*-tilings that are an overall mirror of the *f*-tilings constructed from non-mirrored variants.

2. For a prototile generated by bisecting a polyomino, each bisecting line, which will form the long edge of the prototile, must originate and terminate at corners of the polyomino and pass through the centroid of the polyomino. If the endpoints aren't at corners or pseudo-corners, the short edges will not all be of the same length. The long edges of the prototile must be longer than the short edges or pseudo-edges of the prototile.

3. For a prototile generated by dissecting a polyomino into four equal parts, each dissecting line, which will form one long edge of the prototile, must run from the centroid of the polyomino to a corner. The four dissecting lines are related to each other by rotations of 90° about the centroid. Again, the long edges of the prototile must be longer than the short edges or pseudo-edges of the prototile. For reasons shown below, the number of short edges or pseudo-edges must be even. This rules out any polyomino made up of $4n + 1$ squares [8]. The only other polyominoes with 4-fold symmetry are made up of $4n$ squares, so only these need be considered.

Figure 4 shows candidate polyominoes made up of 1 to 5 squares and prototiles that meet these criteria. Prototiles colored gray allow *f*-tilings, while those colored black do not. Only gray prototiles without tick marks allow true edge-to-edge *f*-tilings.

In the case of prototiles derived from polyominoes with 4-fold rotational symmetry, there are two choices of matching rules for arranging tiles of a given generation around tiles of the next larger generation. This is illustrated in the left side of Figure 5, where the two choices are labeled A and B. The right side of Figure 5 shows why the number of short edges cannot be odd if a single matching rule is used. The fact that there are two long edges for each smaller tile requires an even number of short edges for each larger tile if overlaps are to be avoided.

In the case of polyhexes, two adjacent edges can not lie in a straight line, so all *f*-tilings formed are true edge-to-edge *f*-tilings. Only polyhexes with 2- and 3-fold rotational symmetry need be examined. It can be shown that 6-fold *f*-tilings based on polyhex prototiles will not work [9]. The problem is analogous to that illustrated in Figure 5 for polyominoes; namely, the prototiles always have an odd number of short edges. A table similar to that of Figure 4, but for polyhexes, is found in Reference 9.

In the case of polyiamonds, most candidate prototiles will results in pseudo-edge-to-edge *f*-tilings, such as that in Figures 1 and 2. For polyiamonds, 2-, 3-, and 6-fold *f*-tilings are allowed.

Now that the prototiles have been described in greater detail, we give a number of examples of *f*–tilings constructed from these prototiles. Our first examples have overall 2-fold rotational symmetry. The starting point for an *f*-tiling of this sort is a pair of tiles that form the generating polyomino, polyhex, or polyiamond. These are surrounded by smaller tiles in (pseudo-) edge-to-edge fashion, with the construction process proceeding iteratively. The only option is whether or not the tiles are mirrored between successive generations, but no mirrored variants are considered here due to space limitations.

In Figure 6, we show two examples of *f*-tilings of this sort. The left *f*-tiling is based on the "X" pentomino and is edge-to-edge, with a reduction factor of $1/\sqrt{10}$ between tiles of successive generations. This number is easily obtained by noting that the diagonal is that of three squares in a row and then applying the Pythagorean theorem. (If the squares have edges of length 1, the long edge of the prototile has length $\sqrt{(1^2 + 3^2)}$.) Between successive generations the tiles are rotated by arctan(1/3) ≈ 18.43° (plus multiples of $\pi/2$ as required for fitting a particular edge). The right *f*-tiling in Figure 6 is based on a hexiamond and is pseudo-edge-to-edge, with a reduction factor of $1/\sqrt{7}$.

In Figure 7 we show two *f*-tilings constructed from the same dissection of a 4-hex, using two different matching rules for mating smaller tiles to larger tiles. Both *f*-tilings have 3-fold rotational symmetry and complex fractal boundaries, but the shape of the boundaries is quite different.

For some *f*-tilings, such as that shown at left in Figure 7, holes form that just fill in the infinite limit. From a recreational mathematics standpoint, these holes can be quite interesting. They can be examined by starting with a large polyhex and tiling inward. An example of a relatively complex polyhex-shaped hole (a 14-hex) that persists from generation to generation, though rotated, is shown in Figure 8. The generating 7-hex is shown at left, along with a portion of the *f*-tiling, which illustrates the origin of the hole. At right, the hole is shown filling in through four generations of tiles.

In some cases, an entire *f*-tiling constitutes a supertile that tiles the plane in conventional fashion. A 4-fold example based on a prototile created by dissecting an octomino is shown in Figure 9. A 6-fold example based on a prototile created by dissecting a 36-iamond is shown in Figure 10.

We have presented several examples of fractal tilings (*f*-tilings) based on prototiles derived by dissecting polyominoes, polyhexes, and polyiamonds. There are an infinite number of *f*-tilings of this sort. However, in general they become increasingly less interesting for higher-order polyominoes, polyhexes, and polyiamonds due to the fact that the scaling factor between successive generations becomes more extreme. Fractal tilings that result in recurring polyhex-shaped holes have also been demonstrated, as well as *f*-tilings that form supertiles that in turn tile the infinite mathematical plane.

References

[1] Robert W. Fathauer, *Fractal tilings based on kite- and dart-shaped prototiles*, Computers & Graphics, Vol. 25, pp. 323-331, 2001.

[2] Robert W. Fathauer, *Fractal tilings based on v-shaped prototiles*, Computers & Graphics, Vol. 26, pp. 635-643, 2002.

[3] Robert W. Fathauer, *Self-similar Tilings Based on Prototiles Constructed from Segments of Regular Polygons*, in Proceedings of the 2000 Bridges Conference, edited by Reza Sarhangi, pp. 285-292, 2000.

[4] Robert W. Fathauer, http://members.cox.net/fractalenc/encyclopedia.html.

[5] Bruno Ernst, *The Magic Mirror of M.C. Escher*, Ballantine Books, New York, 1976.

[6] Peter Raedschelders, "Tilings and Other Unusual Escher-Related Prints," in M.C. Escher's Legacy, edited by Doris Schattschneider and Michele Emmer, Springer-Verlag, Berlin, 2003.

[7] Branko Grünbaum and G.C. Shephard, *Tilings and Patterns*, W.H. Freeman, New York, 1987.

[8] The only pentomino with 4-fold rotational symmetry yields prototiles with 3 short edges. Adding 4 squares to this pentomino yields prototiles with either 3 or 5 short pseudo-edges. It can easily be seen that adding four squares to any 4-fold polyomino will either add 2 short pseudo-edges, leave the number of short pseudo-edges unchanged, or subtract 2 short pseudo-edges. The number of short pseudo-edges for prototiles is therefore always odd, which does not allow tiling, as demonstrated in Figure 5.

[9] Robert W. Fathauer, *Fractal Tilings Based on Dissections of Polyhexes*, in Renaissance Banff, Mathematics, Music, Art, Culture Conference Proceedings, 2005, edited by Reza Sarhangi and Robert V. Moody, pp. 427-434, 2005.

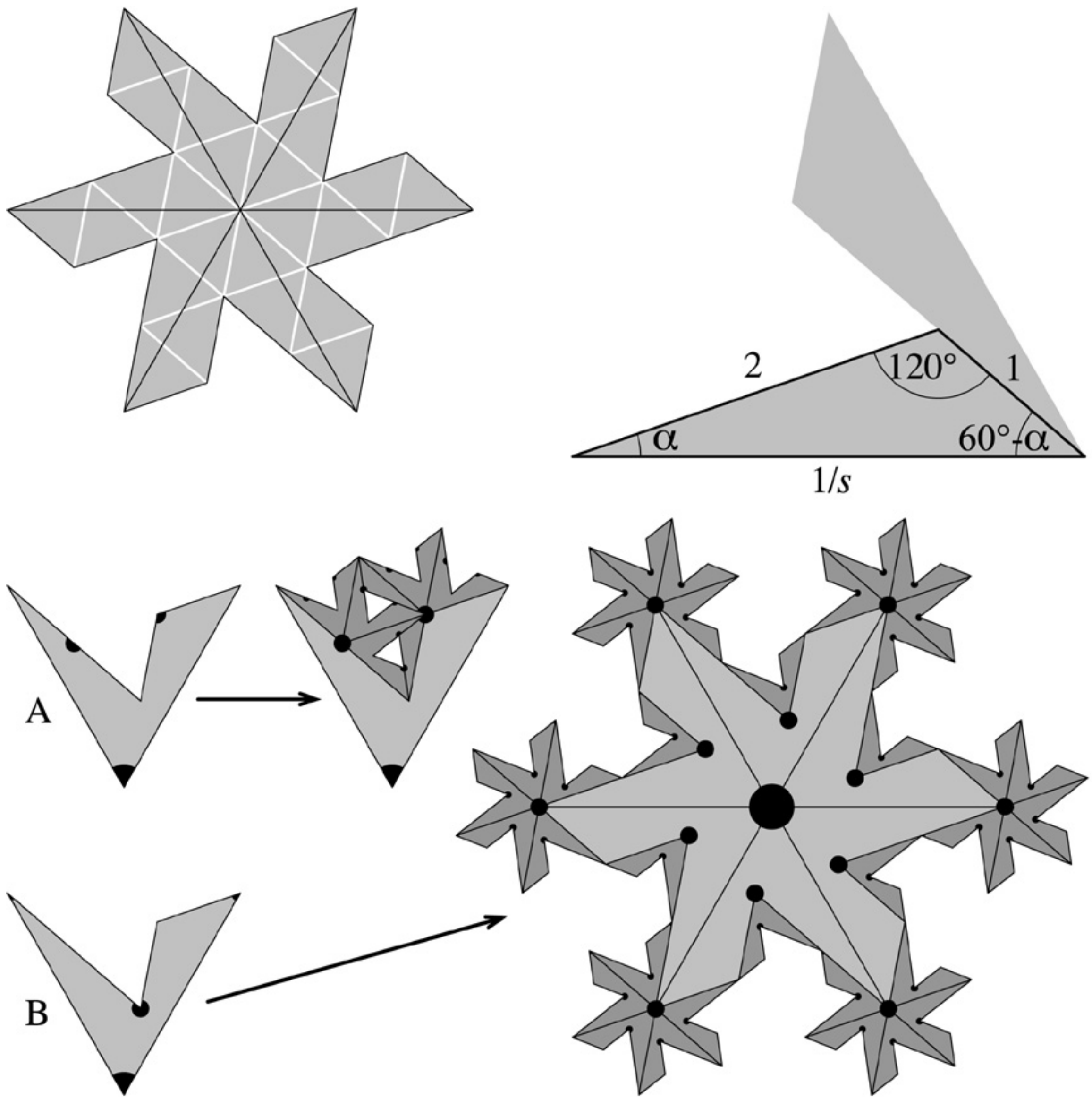

Figure 1 *Derivation of a prototile and its application to the construction of an f-tiling..*

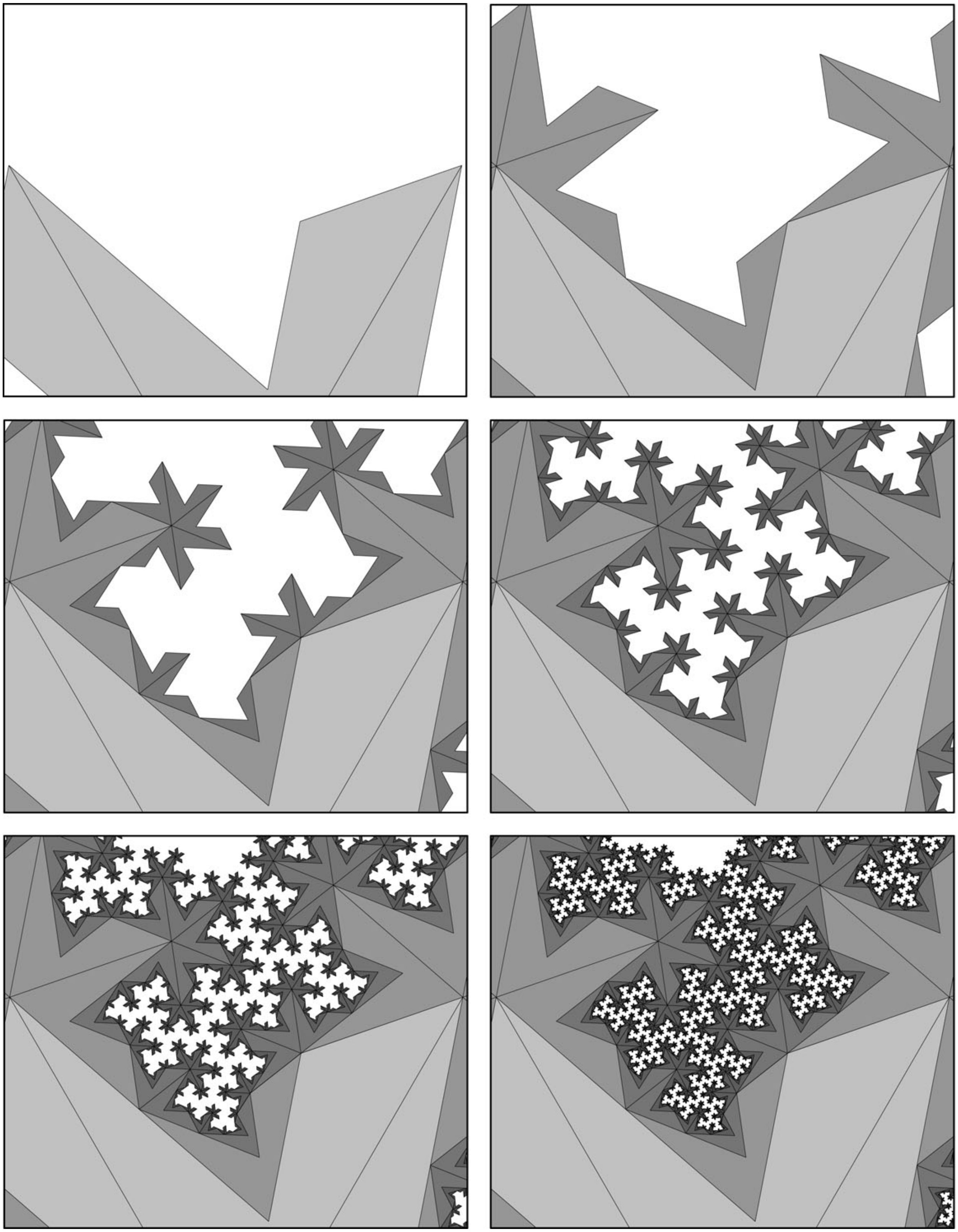

Figure 2 *Construction of an f-tiling using using the prototile from Figure 1 along with Matching Rule B, carried through the first six iterations.*

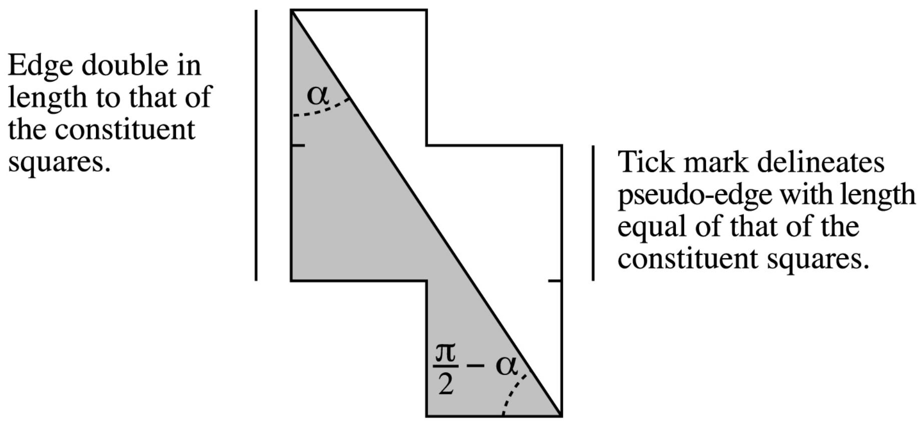

Figure 3 *A tetromino with tick marks indicating pseudo-edges and irrational angle* α*.*

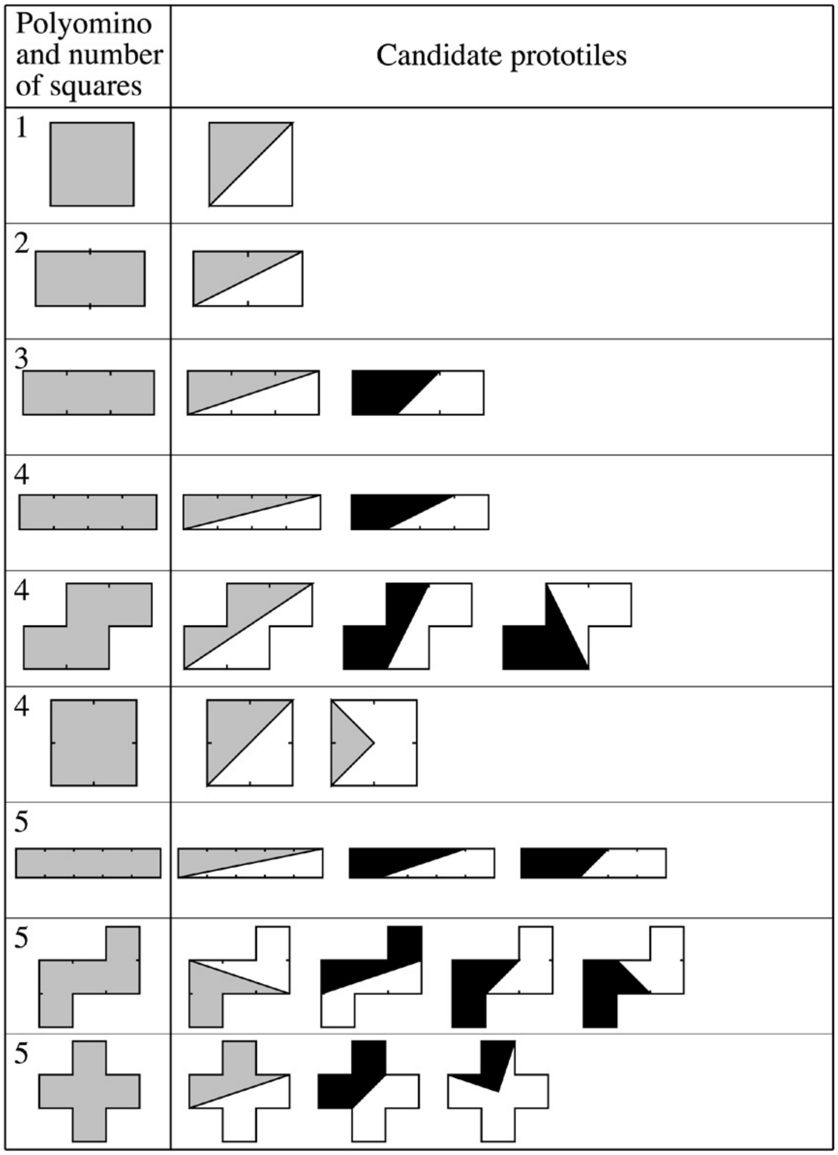

Figure 4 *Candidate prototiles for polyominoes made up of 1 to 5 squares. The gray ones tile, while the black ones do not.*

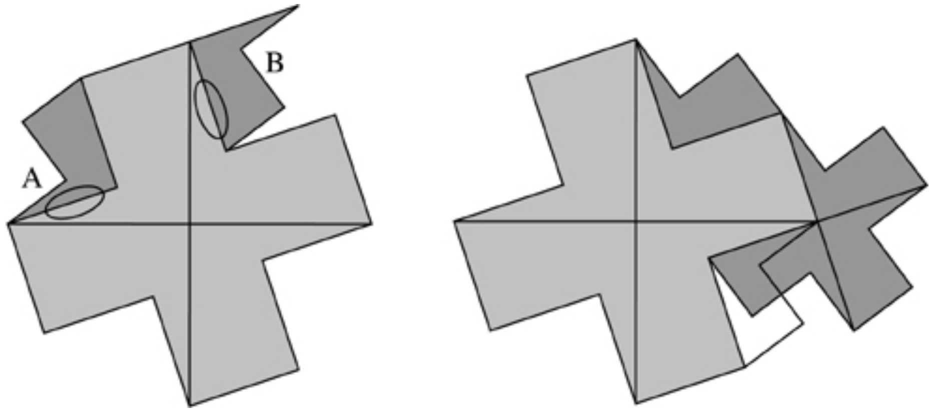

Figure 5 *For prototiles generated from 4-fold polyominoes, there are two choices, A and B, for arranging smaller tiles around large tiles. In order to avoid overlaps using a consistent matching rule, 4-fold prototiles must have an even number of short edges.*

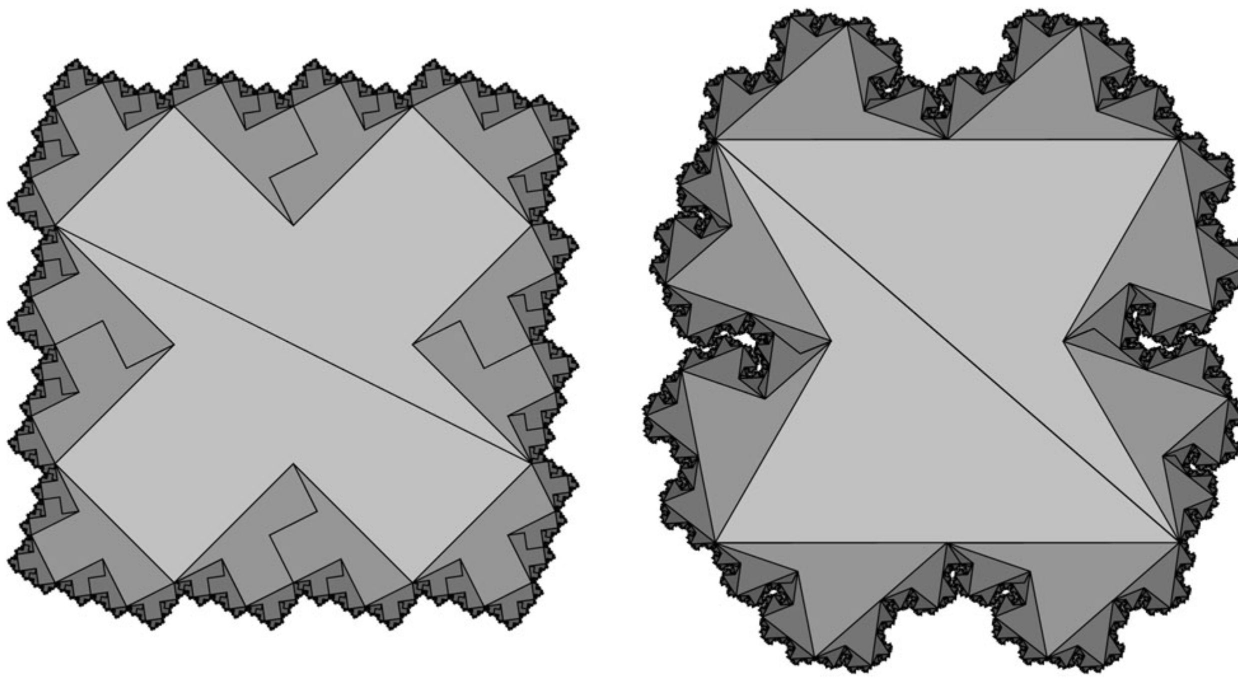

Figure 6 *(Left) A true edge-to-edge f–tiling generated from a prototile based on the "X" pentomino.* (Right) *A pseudo-edge-to-edge f–tiling generated from a prototile based on a hexiamond.*

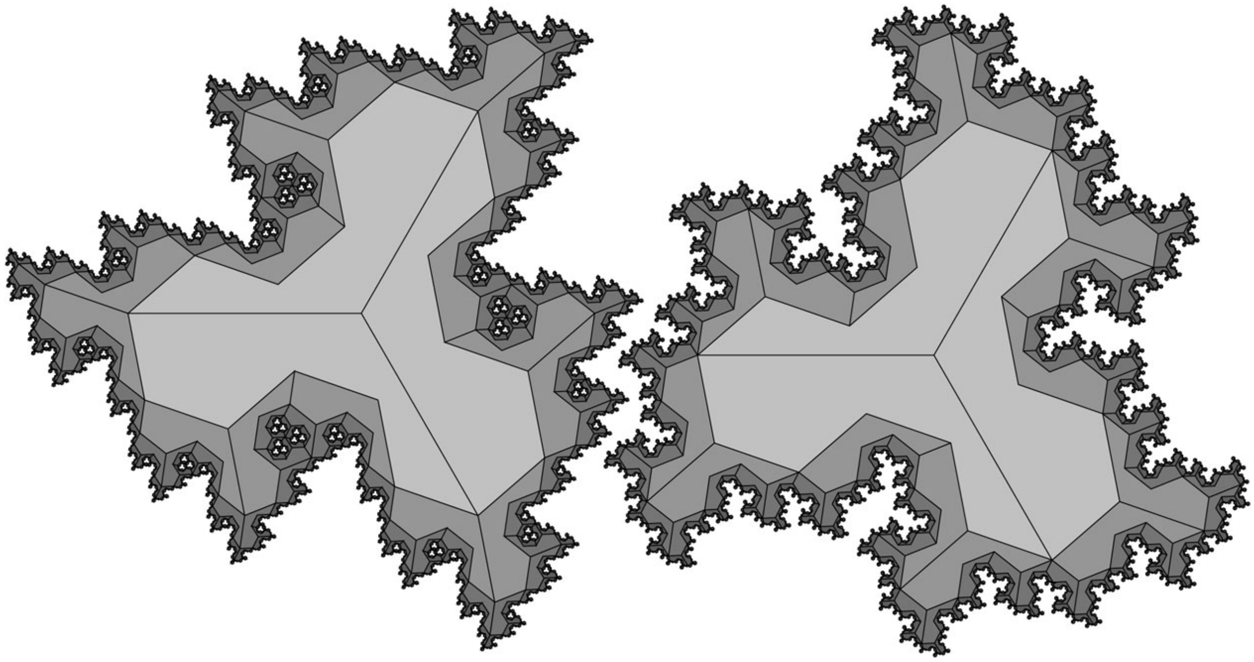

Figure 7 *Two f-tilings generated from a prototile based on a 4-hex, in which the scaling factor between successive generations is approximately 0.3780. Different matching rules are used for the two.*

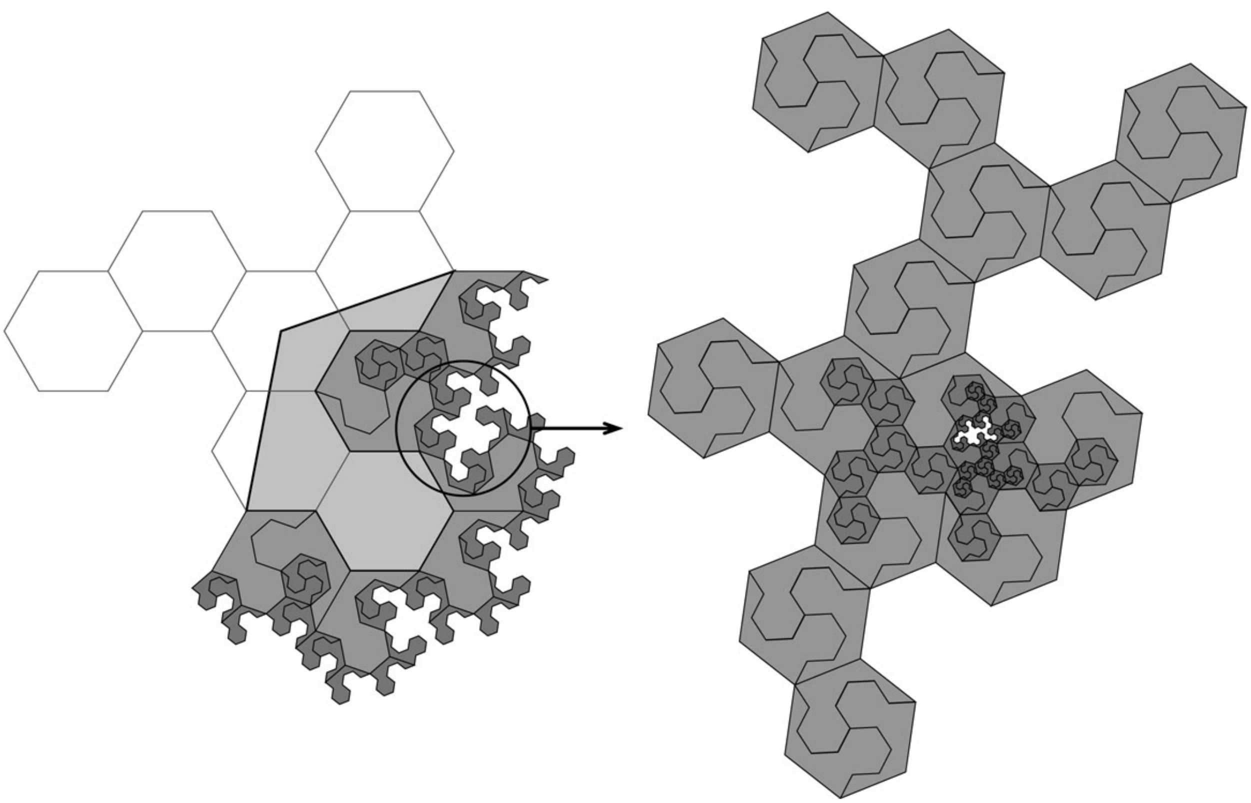

Figure 8 *An example of a 14-hex-shaped hole that persists from generation to generation in an f-tiling based on a dissection of a 7-hex.*

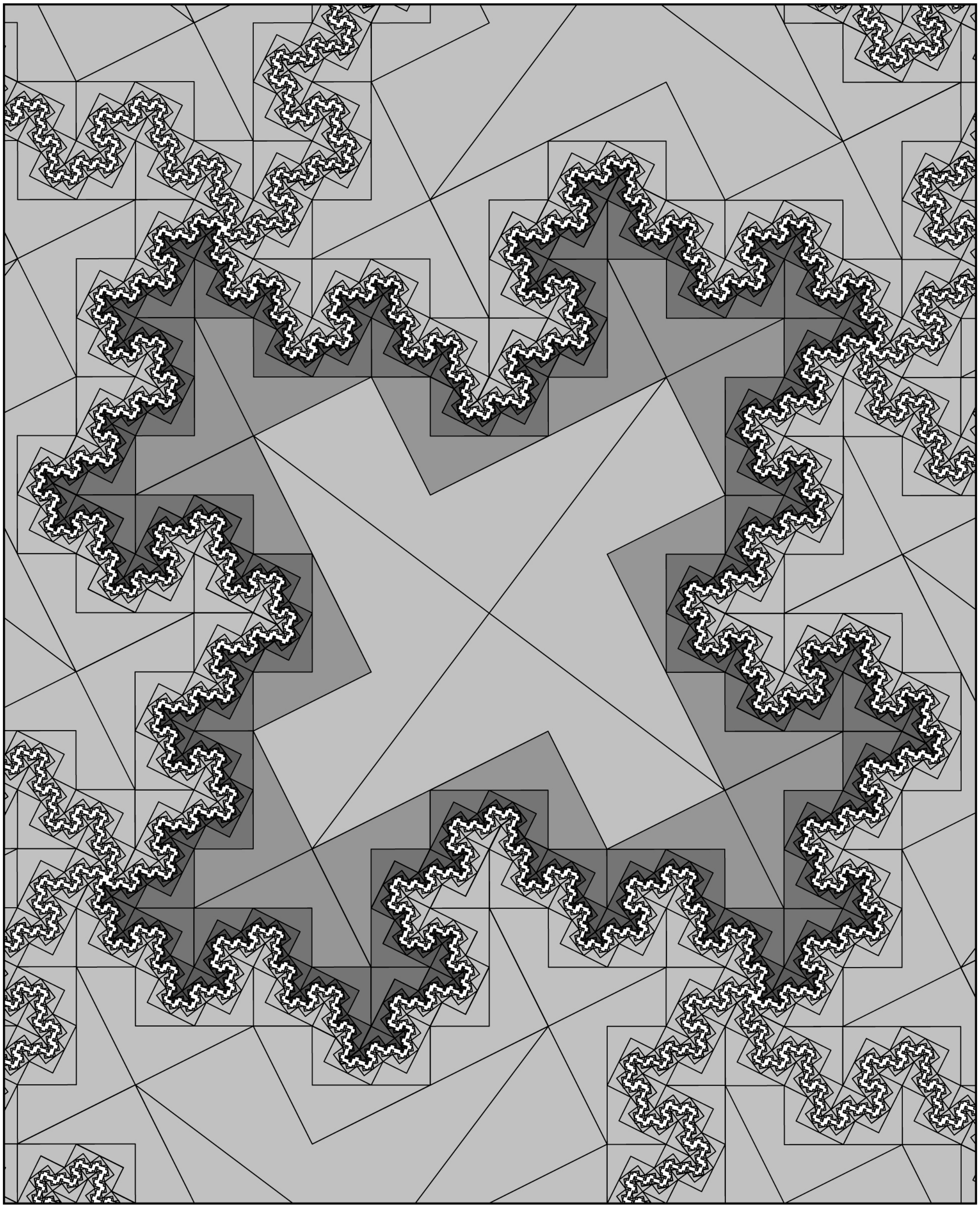

Figure 9 *A plane-filling f–tiling generated from a prototile based on an octomino. The f-tiling was carried through six generations, with each generation a darker shade of gray in the center f-tiling.*

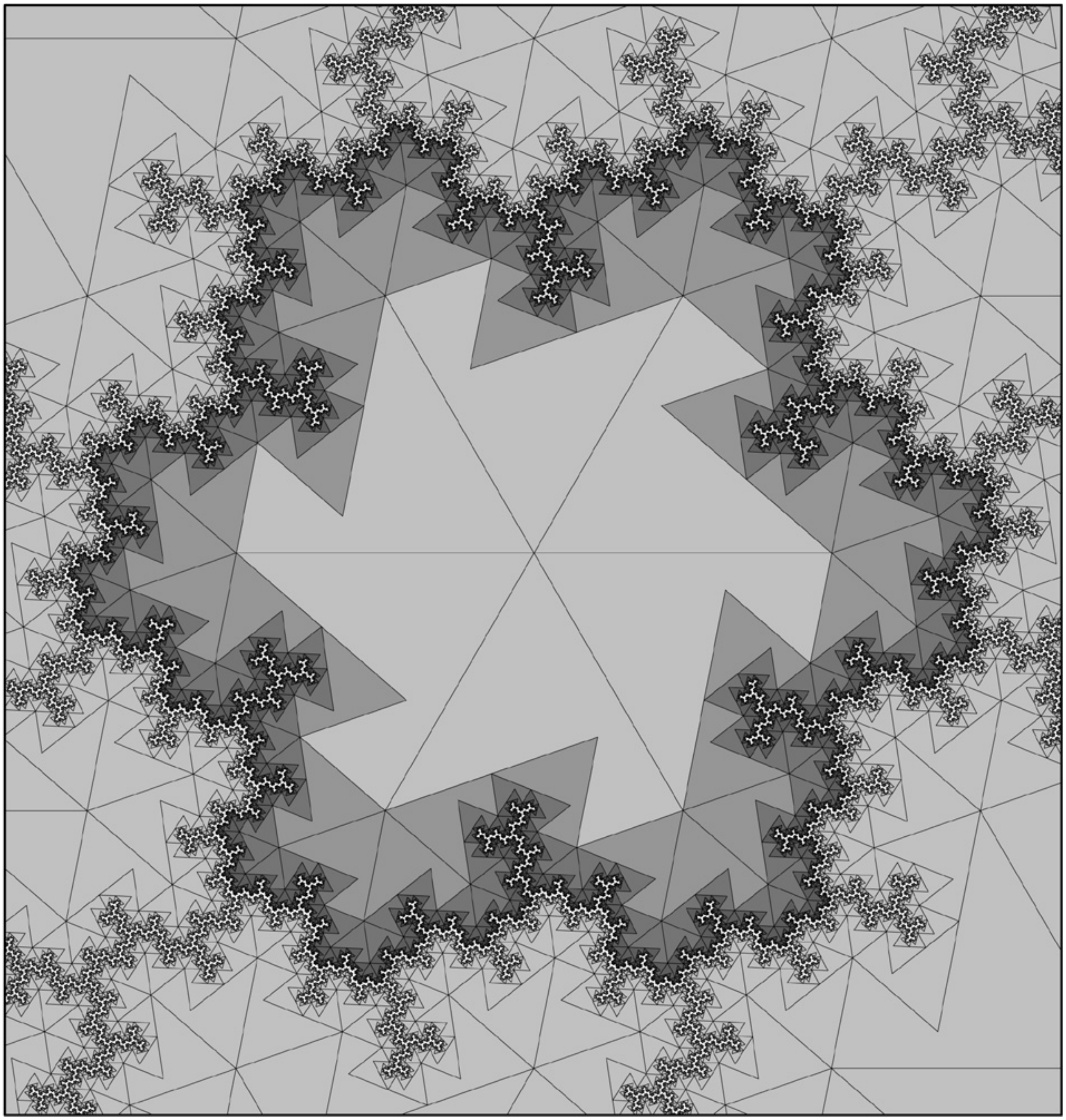

Figure 10 *A plane-filling f–tiling generated from a prototile based on a 36-iamond. The f-tiling was carried through six generations, with each generation a darker shade of gray in the center f-tiling.*